The Chromatic Number of Sparse Graphs

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Erdős Magic Theorem:

For any integers k and g there exists a graph of girth at least g and chromatic number at least k.

Notation:

$$\chi_g(n) = \max\{\chi(G) : |V(G)| = n,\ girth(G) = g\}$$

$$g_k(n) = \max\{girth(G) : \chi(G) = k,\ |G| = n\}$$

1 Triangle-free graphs

Theorem (Ajtai, Komlos, Szemeredi 1980):

There exists a constant c_1 such that any triangle-free graph G on n vertices contains an independent subset of vertices of cardinality $c_1\sqrt{n\log n}$

Theorem (Erdős, Hajnal 1985):

Let $\mathcal{F}$ be a hereditary family of graphs such that for some contionous and non-decreasing function $f(x)$ we have $\alpha(G) \geq f(n)$ for any $G \in \mathcal{F}$ on n vertices. Then

$$\chi(G) \leq \int_2^{|V(G)|} \frac{d(x)}{f(x)} + 2.$$

Corollary:

There exists a greedy algorithm which colors any triangle-free graph on n vertices using $\mathcal{O}(\frac{\sqrt{n}}{\sqrt{\log n}})$

2 On $\chi_g(n)$

Theorem:

- For some constant c, $\chi_g(n) \geq c\,\dfrac{n^{1/(g-2)}}{\ln n}$ (Spencer 1977).
- For odd g, $\chi_g(n) \leq n^{\frac{2}{g-1}} + 1$ (Erdős 1962).
- For even g and some constant c, $\chi_g(n) \leq cn^{2/g}$ (Z. 2007).

3 An extremal problem

Corollary:

For even g

$$\frac{1}{g-2} \le \varliminf \frac{\log \chi_g(n)}{\log n} \le \varlimsup \frac{\log \chi_g(n)}{\log n} \le \frac{2}{g}.$$

Conjecture (Z. 2007):

$$\lim_{n\to\infty} \frac{\log \chi_g(n)}{\log n} \text{ exists.}$$

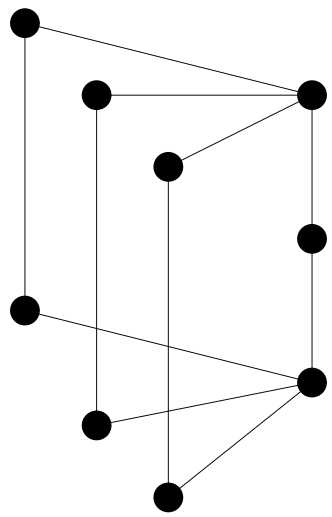
Valid for $g = 3, 4$.

4 The booksize of graphs

Definition:

For any two integers t and k with $0 \leq t \leq k-2$ we denote by $b_{t,k}(G)$ the maximum number of cycles of length k which intersect mutually in a unique path of length t and call it the book number of G with parameter (t,k).

$b_{2,5}$



Theorem (Z. 2007):

(1) There exists a function $c(k)$ for which $\lim_{k\to\infty} c(k) = 1$ such that for any graph G on n vertices, girth $2k+1$ and $j = b_{k,2k+1}(G) + 2$

$$\chi(G) \le c(jn)^{\frac{1}{k+1}} + 2.$$

(2) There exists a function $c(k)$ for which $\lim_{k\to\infty} c(k) = 1$ such that for any graph G on n vertices, girth $2k+1$ and $j = b_{0,2k+1}(G) + 3$

$$\chi(G) \le c(jn)^{\frac{1}{k+1}} + 2.$$

5 The even girth of graphs

Theorem (Z. 2007):

Let the smallest length of an even cycle in G be $2k+2$. Let also $j = 2\prod_{i=1}^{k}(b_{0,2i+1}(G)+2)$. Then for some constant $c = c(k)$ for which $\lim_{k\to\infty} c(k) = 1$

$$\chi(G) \leq c(jn)^{\frac{1}{k+1}} + 2.$$

6 One more result

For any family $\mathcal{F}$ of graphs define $\chi_5(n, \mathcal{F})$ as the maximum chromatic number of any graph of girth at least five in $\mathcal{F}$.

Theorem (Z. 2007):

Let for some function $f(n) = o(n)$ the family $\mathcal{F}$ be defined as either $\{G : b_{2,5}(G) \leq n^{f(n)}\}$ or $\{G : b_{0,5}(G) \leq n^{f(n)}\}$ where n denotes the order of graph. Then

$$\lim_{n\to\infty} \frac{\log \chi_5(n, \mathcal{F})}{\log n} = 1/3.$$

7 Some lower bounds for chromatic number

Definition: A family $\mathcal{F}$ of graphs is called a color-bounded family if for some function $f(x)$ and any G from the family one has $\chi(G) \geq f(col(G))$.

Theorem (Markossian, Gasparian, Reed 1996):

Let G be a graph without any even-hole. Then

$$\chi(G) \geq \frac{col(G) - 1}{2}.$$

Theorem (Z. 2008):

Let T be an arbitrary tree on k vertices and G a $(K_{2,t}, T)$-free graph. Let also $\lambda = 2(k-2)(t-1)$. Then

$$\chi(G) \geq \frac{d(G)}{\lambda} + 1.$$

Theorem (Z. 2008):

Let the maximum even-hole of a graph G be k. Then

$$\chi(G) \geq \frac{d(G)}{k} + 1.$$

Theorem (Z. 2008):

Let G be a $K_{1,t+1}$-free graph. Then

$$\frac{\Delta(G)+t}{t} \leq \chi(G).$$

Theorem (Z. 2008):

For any k there exists a bipartite graph G so that $\delta(G) > k$ and $girth(G) > k$.