Positive scalar — lectures in Teheran February 2008

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Abstract

In these notes, we address the question if a given smooth manifold M admits a Riemannian metric with positive scalar curvature. If so, we also introduce the attempts to classify all such metrics.

The principal tool to invistigate these problems is the Lichnerowicz formula, which connects these problems to index theory — in particular to the index of the Dirac operator. A decisive role in the calculation of these invariants is played by the Atiya-Singer index theorem. In this realm, it is useful to use "higher" variants of these invariants to get optimal information, and corresponding extensions of the Atiyah-Singer index theorem. In particular, we will use the K-theory of C^*-algebras and index obstructions in these K-theory groups.

For the classification results we rely on the Atiyah-Patodi-Singer index theorem and the calculation of eta-invariants.

If time permits, we will also discuss briefly obstructions to positive scalar curvature of Schoen-Yau, using the geometry of stable minimal hypersurfaces and geometric measure theory.

1 Organisation of the talks

(1) First lecture: Repetition of foundational material

- Riemannian geometry of positive scalar curvature (repetition)
- constructions of positive scalar curvature metrics
- spin structures and Dirac operator (repetition)

- Weitzenb"ock formula (repetition)
- Atiyah-Singer index theorem and first obstructions to positive scalar curvature: $K3$-surfaces and the Bott-manifold do not admit metrics with positive scalar curvature

(2) Second lecture: K-theory obstructions to positive scalar curvature

- Family index theory
- C^*-algebras and their K-theory
- Mishchenko-Fomenko index theorem
- Refined obstructions to positive scalar curvature (Rosenberg index)
- The Gromov-Lawson-Rosenberg conjecture about positive scalar curvature

(3) Third lecture: the power of the Rosenberg index

- Enlargeability
- The Gromov-Lawson obstruction to positive scalar curvature: existence of negative sectional curvature prevents positive scalar curvature
- codimension 2 obstructions to break the symmetry (a reinterpretation of another result of Gromov-Lawson in the context of higher index theory)

(4) Fourth lecture: Stephan Stolz proof of the stable Gromov-Lawson-Rosenberg conjecture

- the stable Gromov-Lawson-Rosenberg conjecture
- Kasparov's KK-theory
- index in KK-theory
- K-homology; the Baum-Connes conjecture
- HP^2-bundles and the corresponding cohomology theory
- Stephan Stolz' proof of the stable Gromov-Lawson-Rosenberg conjecture
- Vanishing results for L^2-rho invariants (for the concept of rho- and eta-invariants, compare Paolo Piazza's talks)

(5) Fifth lecture: Classification of metrics of positive scalar curvature

- The Atiyah-Patodi-Singer index theorem (repetition)
- equivalence relations on the space of metrics of positive scalar curvature: connectedness, concordance, cobordism
- action of the diffeomorphism group, construction of a moduli space

- construction of many connected components (even concordance classes) on the space of metrics of positive scalar curvature
- Piazza-Schick: construction of many cobordism classes, provided the fundamental group contains torsion (using L^2-rho invariants, introduced by Paolo Piazza)
- non-trivial fundamental group of the space of metrics of positive scalar curvature

(6) Sixth lecture: minimal surface obstructions to positive scalar curvature (after Schoen-Yau)

- (stable) minimal hypersurfaces
- geometric measure theory force the existence of minimal surfaces
- positive scalar curvature and minimal surfaces (Schoen-Yau)
- Counterexamples to the (unstable) Gromov-Lawson-Rosenberg conjecture