Spherical Designs and Association Schemes
Versus
Euclidean Designs and Coherent Configurations

Eiichi Bannai
Kyushu University
—joint with Etsuko Bannai —

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$$\begin{tikzcd}
\textbf{Spherical designs} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \textbf{Association schemes} \arrow[d, leftrightarrow] \\
\textbf{Euclidean designs} \arrow[r, leftrightarrow] & \textbf{Coherent configurations}
\end{tikzcd}$$

Spherical t-design = approximating the sphere by a finite set (w.r.t. the integrals of polynomials)

Delsarte-Goethals-Seidel 1977

$$S^{n-1} = \{(x_1, \ldots, x_n) \in R^n \mid x_1^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^n$$

$X \subset S^{n-1}$, $|X| < \infty$, is a spherical t-design

$\Longleftrightarrow$

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x) d\sigma(x) = \frac{1}{|X|} \sum_{x \in X} f(x)$$

for $\forall f(x) = f(x_1, \ldots, x_n)$, polynomials of degree $\leq t$.

Here, $|S^{n-1}|$ =the area of S^{n-1}, and the integral in the LHS is the usual surface integral on S^{n-1}

More generally, for $r > 0$,
$S^{n-1}(r) = \{(x_1, \ldots, x_n) \in R^n \mid x_1^2 + \cdots + x_n^2 = r^2\} \subset \mathbb{R}^n$.
$X \subset S^{n-1}(r)$, $|X| < \infty$, is a spherical t-design

$$\Longleftrightarrow \frac{1}{r}X \subset S^{n-1} \text{ is a spherical } t\text{-design}$$

$$\Longleftrightarrow \frac{1}{|S^{n-1}(r)|}\int_{S^{n-1}(r)} f(x)d\sigma(x) = \frac{1}{|X|}\sum_{x\in X} f(x)$$

for $\forall f(x) = f(x_1, \ldots, x_n)$, polynomials of degree $\leq t$.

Equivalent definitions of spherical t- design

$X \subset S^{n-1}$ is a spherical t-design

$\Longleftrightarrow$

$\sum_{x \in X} f(x) = 0$, $\forall f(x)$, homogeneous harmonic polynomials of degree $1, 2, \ldots, t$.

$\Longleftrightarrow$

All kinds of moments of degree $\leq t$ of X are invariant under any orthogonal transformation.
More precisely

$$\sum_{x \in X} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} = \sum_{x \in \sigma(X)} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

$\lambda_1, \ldots, \lambda_n \geq 0, \quad \lambda_1 + \cdots + \lambda_n \leq t, \quad {}^{\forall}\sigma \in O(n)$

Facts on spherical t-designs

1. X is a t-design $\Longrightarrow$ X is a i-design ${}^{\forall} i \leq t$.
2. X is a t-design $\Longrightarrow$ $\sigma(X)$ is a t-design ${}^{\forall} \sigma \in O(n)$.
3. $X_1,\ X_2$ $(X_1 \cap X_2 = \emptyset)$ are t-designs $\Longrightarrow$ $X_1 \cup X_2$ is a t-design.

Lower bounds (Fisher type inequality)

(Delsarte-Goethals-Seidel 1977)

X is a t-design in $S^{n-1} \subset \mathbb{R}^n$

$$t = 2e \Longrightarrow |X| \geq \binom{n-1+e}{e} + \binom{n-1+e-1}{e-1}$$

$$t = 2e+1 \Longrightarrow |X| \geq 2\binom{n-1+e}{e}$$

If " $=$ " holds, then X is a spherical tight t-design.

Examples

• Vertices of a regular $(t+1)$-gon on the cirlle S^1 form a t-design, and it is a tight t-design.

• Vertices of a regular polyhedron in $S^2 \subset \mathbb{R}^3$ form a spherical t-design.

regular polyhedron	no. of vertices	t	tight
simplex	4	2	yes
cube	6	3	yes
octahedron	8	3	no
icosahedron	12	5	yes
dodecahedron	20	5	no

• Many good examples of spherical t-designs are obtained as orbits of finite subgroups $G \subset O(n)$

$X = \{g(x_0) \mid g \in G\} \subset S^{n-1}$ for a fixed $x_0 \in S^{n-1}$

• Many good examples of spherical t-designs are obtained as shells of lattices $L \subset \mathbb{R}^n$

$X = L_r = \{x \in L \mid \|x\|^2 = r^2\}$

• $L = E_8$-lattice $\subset \mathbb{R}^8$

$G = Aut(L) = W(E_8) \subset O(8)$.

All the orbits of $G = W(E_8)$ are spherical 7-designs.

(Some of them are 11-designs.)

All the shells of E_8 lattice L are 7-designs.

(It is an open question whether any of them is an 8-design. This is equivalent to Lehmer's conjecture.)

• $L =$ Leech lattice $\subset \mathbb{R}^{24}$
$G = Aut(L) =$ Conway $\cdot\, 0 \subset O(24)$.
All the orbits of G are spherical 11-designs.
(Some of them are 15-designs.)
All the shells of Leech lattice L are 11-designs.
(It is an open question whether any of them is a 12-design.)

• As far as the known examples with $n \geq 3$ are concerned, those obtained as orbits of $G \subset O(n)$ are at most 19-designs, and those obtained as shells of lattices in $\mathbb{R}^n$ are at most 11-designs.

So it is an interesting question whether any 12-design is obtained as a shell of a lattice.

Also, it is an interesting question whether any t-design with arbitrary large t are obtained as orbits of finite groups in $O(n)$.

• Theorem (Seymour-Zaslavsky 1984) for any t and for any n, spherical t-design X on S^{n-1} exists.

• Explicit constructions of spherical t-design X for large t on S^{n-1} for $n \geq 3$ are difficult in general. (cf. G. Kuperberg 2006 for $n = 3$)

• **Tight spherical t-designs on S^{n-1} are classified (up to orthogonal transformations) except for $t = 4, 5, 7$ (Bannai-Damerell 1979, 1980).**

If $n \geq 3$, then

$t = 1 \Longrightarrow \quad |X| = 2$, **a pair of antipodal points.**

$t = 2 \Longrightarrow \quad |X| = n + 1$, **regular simplex.**

$t = 3 \Longrightarrow \quad |X| = 2n$, **cross polytope (gen. octahedron).**

$t = 11 \Longrightarrow \quad n = 24,\ |X| = 196560,\ X =$ **the set of min. vectors of Leech lattice in** $\mathbb{R}^{24}$

$t = 4 \Longrightarrow \quad n = (2k+1)^2 - 3$

$t = 5 \Longrightarrow \quad n = 3$ **or** $(2k+1)^2 - 2$

$t = 7 \Longrightarrow \quad n = 3d^2 - 4$

Bannai-Munemasa-Venkov (2004) obtained more nonexistence results for $t = 4, 5, 7$.

Association schemes and coherent configurations

Association scheme $(X, \{R_i\}_{i \in I})$ is a pair of a finite set X and a set of relations $\{R_i\}_{i \in I}$ on X satisfying certain axioms.

Coherent configuration is a more general concept (than association scheme) defined as follows.

coherent configurations

X : a finite set $R_1,\ R_2, \ldots, R_l \subset X \times X$.

> If the following conditions (1)～(4) are satisfied, then $\mathfrak{X} = (X, \{R_i\}_{1 \le i \le l})$ is a coherent configuration

1. $X \times X = R_1 \cup R_2 \cup \cdots \cup R_l$ is a partition.
2. $^{\exists}p$ s.t. $1 \le p < l$, $R_1 \cup \cdots \cup R_p = \{(x, x) \mid x \in X\}$.
3. For each i, $^{\exists}i'$ such that $^{t}R_i = R_{i'}$, $1 \le i' \le l$, (where $^{t}R_i := \{(x, y) \mid (y, x) \in R_i\}$)
4. For each i, j, k, $|\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|$ is a constant on $(x, y) \in R_k$ (depends only on i, j, k). (We denote it by $p_{i,j}^{k}$.)

Association schemes are special cases of coherent configurations with $p = 1$, i.e.,

$$\{(x, x) \mid x \in X\} = R_1.$$

• Coherent configuration was defined by D. G. Higman in 1970, and is a combinatorial axiomatization of general (not necessarily transitive) finite permutation groups.

• Association scheme is a combinatorial axiomatization of transitive finite permutation groups.

• Important classes of association schemes:
P-polynomial association schemes,
Q-polynomial association schemes,
P- and Q-polynomial association schemes.

s-distance set on S^{n-1}

Let $X \subset S^{n-1}$ be a finite set. Define

$$A(X) = \{x \cdot y \mid x \neq y \in X\}$$

X is called an s-distance set if $|A(X)| = s$.

• Theorem (Delsarte-Goethals-Seidel 1977)
Let $X \subset S^{n-1}$ be a finite set which is a t-design and an s-distance set. Then the followings hold:

1. $t \leq 2s$.
2. $t = 2s \iff X$ is a tight $2s$-design.
3. $t = 2s - 1$ and X is antipodal $\iff X$ is a tight $(2s-1)$-design.

Moreover, we have

4. $\boxed{t \geq 2s - 2} \Longrightarrow$ $\boxed{\begin{array}{l}(X, \{R_i\}_{0\leq i\leq s}) \\ \text{is a Q-polynomial} \\ \text{scheme.}\end{array}}$

Here we define

$$A(X) = \{\alpha_1, \ldots, \alpha_s \mid -1 \leq \alpha_i < 1\}.$$

$$R_i = \{(x, y) \in X \times X \mid x \cdot y = \alpha_i\}, \quad (1 \leq i \leq s)$$

$$R_0 = \{(x, x) \mid x \in X\}$$

Here we use the notations $R_0, \ldots, R_s$ **instead of** $R_1, \ldots, R_{s+1}$

- **Remark (B-B):**

$\boxed{\begin{array}{l} t \geq 2s - 3 \text{ and} \\ X \text{ is antipodal}\end{array}} \Longrightarrow \boxed{\begin{array}{l}(X, \{R_i\}_{0\leq i\leq s}) \\ \text{is a Q-polynomial} \\ \text{scheme.}\end{array}}$

Euclidean t-designs

A two step generalization of spherical t-designs, that is X has a weight w and X is in $\mathbb{R}^n$ (not necessarily on S^{n-1}.)

Notation:

$X \subset \mathbb{R}^n$, a finite set

$\{\|x\| \mid x \in X\} = \{r_1, \ldots, r_p\}$,

$S_i = \{x \in \mathbb{R}^n \mid \|x\| = r_i\}$, $X_i = S_i \cap X$ $(1 \le i \le p)$.

We say X is supported by $S = \cup_{i=1}^{p} S_i$.

$$\varepsilon_S = \begin{cases} 0 & \text{if } 0 \notin S \\ 1 & \text{otherwise.} \end{cases}$$

$w: X \longrightarrow \mathbb{R}_{>0}$, a weight function

$w(X_i) = \sum_{x \in X_i} w(x)$,

$|S^{n-1}| = \int_{S^{n-1}} d\sigma(x)$, $|S_i| = \int_{S_i} d\sigma_i(x)$,

If $r_i = 0$, then $\frac{1}{|S_i|}\int_{S_i} f(x)d\sigma_i(x) = f(0)$ for $^\forall f(x) \in \mathcal{P}(n)$,
$|S_i| = {r_i}^{n-1}|S^{n-1}|$ for $r_i > 0$.

Definition(Neumaier-Seidel, 1988)
(X, w) is a Euclidean t-design if

$$\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|}\int_{S_i} f(x)d\sigma_i(x) = \sum_{x\in X} w(x)f(x)$$

for any polynomial $f(x)$ of degree at most t, wehre $w(X_i) = \sum_{x\in X_i} w(x)$.

Remark:
$p = 1$, $X \neq \{0\}$, $w(x) \equiv 1$, $\Longrightarrow$ Spherical t-designs.

Equivalent definitions of Euclidean t-design

1. (X, w) is a Euclidean t-design.
2. The following equation holds
$$\sum_{x \in X} w(x) \|x\|^{2j} \varphi_l(x) = 0$$
for any homogeneous harmonic polynomial φ_l of degree l, where l and j are integers satisfying $1 \leq l \leq t$ and $0 \leq j \leq \frac{t-l}{2}$.
3. All kinds of moments of degree $\leq t$ of X are invariant under any orthogonal transformation.

Namely

$$\sum_{x \in X} w(x) f(x) = \sum_{x \in X} w(x) f(\sigma(x))$$

holds for any polynomial f of degree $\leq t$ and $\sigma \in O(n)$.

Natural lower bounds

Theorem (Möller 1976)

Let $X \subset \mathbb{R}^n$ be a finite set and w be a positive weight function on X.

1. (X, w): Euclidean $2e$-design $\Longrightarrow |X| \geq \dim(\mathcal{P}_e(n)|_S)$.
2. (X, w): Euclidean $(2e+1)$-design.
   (a) e odd, or e even and $0 \notin X \Longrightarrow |X| \geq 2\dim(\mathcal{P}_e^*(n)|_S)$.
   (b) e even and $0 \in X \Longrightarrow |X| \geq 2\dim(\mathcal{P}_e^*(n)|S) - 1$,

where $\mathcal{P}_e(n) = \oplus_{i=0}^{e}\mathrm{Hom}_i(n)$, $\mathcal{P}_e^*(n) = \oplus_{i=0}^{[\frac{e}{2}]}\mathrm{Hom}_{e-2i}(n)$, where $\mathrm{Hom}_i(n)$ is the space of homogeneous polynomials of degree i, and $S = S_1 \cup \cdots \cup S_p$ (= the set of concentric spheres intersecting with X).

Tight designs

If " = " holds in the previous page, then (X, w) is a tight t-design on p concentric spheres

Moreover if

(1) $\dim(\mathcal{P}_e(n)|_S) = \dim(\mathcal{P}_e(n))$ (for $t = 2e$),

or

(2) $\dim(\mathcal{P}^*_e(n)|_S) = \dim(\mathcal{P}^*_e(n))$
(for $t = 2e + 1$)

holds, then (X, w) is a Euclidean tight t-design

Some more notation

(X, w): Euclidean t-design in $\mathbb{R}^n$.

For any $X_\lambda, X_\mu \neq \{0\}$, we define

$$A(X_\lambda, X_\mu) := \left\{ \frac{x \cdot y}{\|x\| \|y\|} \mid x \in X_\lambda, y \in X_\mu, x \neq y \right\}.$$

Let $s_{\lambda,\mu} := |A(X_\lambda, X_\mu)|$,
$A(X_\lambda, X_\mu) = \{\alpha_{\lambda,\mu}^{(u)} \mid u = 1, \ldots, s_{\lambda,\mu}\}$, $\alpha_{\lambda,\lambda}^{(0)} := 1$.

(Then clearly $A(X_\lambda, X_\mu) = A(X_\mu, X_\lambda)$,
$s_{\lambda,\mu} = s_{\mu,\lambda}$ and $\alpha_{\lambda,\mu}^{(u)} = \alpha_{\mu,\lambda}^{(u)}$.)

The following results (Theorem A $\sim$ Theorem E) are the main theorems of this talk.

Theorem A

(X, w): a Euclidean t-design, $w(x) \equiv w_\nu$ for any $x \in X_\nu$. and one of the following (1) or (2) holds.

1. If $s_{\lambda,\nu} + s_{\nu,\mu} \leq t - 2(p - \varepsilon_S - 2)$ holds for any λ, ν and μ with $1 \leq \lambda, \nu, \mu \leq p$.
2. If X is antipodal and

   $s_{\lambda,\nu} + s_{\nu,\mu} - \delta_{\lambda,\nu} - \delta_{\nu,\mu} \leq t - 2(p - \varepsilon_S - 2)$

   holds for any λ, ν and μ satisfying

   $1 \leq \lambda, \nu, \mu \leq p$.

Then X has the structure of a coherent configuration.

In other words, for $(\boldsymbol{x}, \boldsymbol{y}) \in X_\lambda \times X_\mu$**, with** $\boldsymbol{x} \cdot \boldsymbol{y} = \alpha^{(k)}_{\lambda,\mu}$**,**

$$|\{\boldsymbol{z} \in X_\nu \mid \boldsymbol{x} \cdot \boldsymbol{z} = \alpha^{(i)}_{\lambda,\nu},\ \boldsymbol{z} \cdot \boldsymbol{y} = \alpha^{(j)}_{\nu,\mu}\}|$$

depends only on λ, ν, μ, i, j, k**. (Here** $1 \leq \lambda, \nu, \mu \leq p$**, and** $1 - \delta_{\lambda,\nu} \leq i \leq s_{\lambda,\nu}$**,** $1 - \delta_{\nu,\mu} \leq j \leq s_{\nu,\mu}$**, and** $1 - \delta_{\lambda,\mu} \leq k \leq s_{\lambda,\mu}$**.)**

Theorem B

Let (X, w) be a tight Euclidean t-design supported by 2 concentric spheres. Then X has the structure of a coherent configuration.

Towards the classification of Euclidean 4-designs on 2 concentric spheres having the structures of coherent configurations.

Theorem C

(X, w): a Euclidean 4-design in $\mathbb{R}^n$ on **2** concentric spheres. $0 \notin X$ and w is constant on each X_λ, $s_{\lambda,\mu} \leq 2$ $(\lambda, \mu = 1, 2)$. Then X has the structure of a coherent configuration and the following holds.

(1) $s_{1,2} = 2$.

(2) (X, w) is a tight Euclidean 4-design or similar to one of the Euclidean 4-design having the parameters given in (i) and (ii).

(i) $n = 2$**,**

$X_1 = \{\pm(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}),\ \pm(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}})\}$**,**

$X_2 = \{(\pm r_2,\ 0),\ (0,\ \pm r_2)\}$**,**

$w(\boldsymbol{x}) = 1$ **on** X_1**,** $w(\boldsymbol{x}) = r_2^{-4}$ **on** X_2**,**

r_2**: any positive real number satisfying** $r_2 \neq 1$**.**

(ii) $n = (2k-1)^2 - 4$**,**

$|X_1| = 2(2k+1)(k-1)^3$**,**

$|X_2| = 2k^3(2k-3)$**,**

$A(X_1) = \{\frac{k-2}{k(2k-3)},\ -\frac{1}{2k-3}\}$**,**

$A(X_2) = \{\frac{1}{2k+1}, -\frac{k+1}{(k-1)(2k+1)}\}$**,**

$A(X_1, X_2) = \{\frac{1}{\sqrt{n}},\ -\frac{1}{\sqrt{n}}\}$**,**

$r_1 = 1$**,** $w_1 = 1$**,**

$w_2 = \frac{(2k+1)^2(k-1)^4}{(2k-3)^2k^4} r_2^{-4}$,

where k **is any integer satisfying** $k \geq 2$ **and** r_2 **is any positive real number satisfying** $r_2 \neq 1$**.**

The intersection numbers of the corresponding coherent configurations are given in the Appendix I.

Theorem D

A Euclidean 4-design in $\mathbb{R}^n$ having the parameters given in Theorem C (2) (ii) exists if and only if a tight spherical 4-design on $S^n \subset \mathbb{R}^{n+1}$ exists.

<u>Theorem E</u> The following is a series of feasible parameters for tight Euclidean 4-design in $\mathbb{R}^n$.

$$n = (6k-3)^2 - 3,$$

$$|X_1| = (6k^2-6k+1)(36k^2-36k+7),$$

$$|X_2| = 3(36k^2-36k+7)(2k-1)^2,$$

$$A(X_1) = \left\{ \frac{18k^2-27k+8}{6(9k^2-9k+1)(2k-1)}, \quad -\frac{18k^2-9k-1}{6(9k^2-9k+1)(2k-1)} \right\},$$

$$A(X_2) = \left\{ \frac{36k^3-54k^2+25k-4}{2(6k^2-6k+1)(18k^2-18k+5)}, \quad -\frac{36k^3-54k^2+25k-3}{2(6k^2-6k+1)(18k^2-18k+5)} \right\},$$

$$A(X_1, X_2) = \left\{ \sqrt{\frac{36k^2-36k+4}{(36k^2-36k+6)(36k^2-36k+10)}}, -\sqrt{\frac{36k^2-36k+10}{(36k^2-36k+6)(36k^2-36k+4)}} \right\},$$

$$r_1 = 1, \qquad r_2 = \sqrt{\frac{3(18k^2-18k+5)(6k^2-6k+1)}{9k^2-9k+1}},$$

$$w_1 = 1, \qquad w_2 = \frac{1}{81(2k-1)^4}.$$

(2) If $2 \leq n \leq 15^2 - 3$, then tight Euclidean 4-design supported by 2 concentric spheres is similar to one of the examples given in Theorem I, II and III in the paper by Etsuko Bannai (2009) or to the one of those having the parameters given in Theorem E .

The intersection numbers of the corresponding coherent configurations are given in the Appendix II.

Examples of tight Euclidean 4-designs on 2 concentric spheres(Etsuko Bannai 2009)

Theorem I. $|X_1| = n + 1$**.**

n	$\|X_1\|$	$\|X_2\|$	r_1	r_2	$A(X_1)$	$A(X_2)$	$A(X_1, X_2)$	w_1	w_2
2	3	3	1	$r \neq 1$	$-\frac{1}{2}$	$-\frac{1}{2}r^2$	$\frac{1}{2}r,\ \ -r$	1	$\frac{1}{r^3}$
4	5	10	1	$\frac{1}{\sqrt{6}}$	$-\frac{1}{4}$	$\frac{1}{36},\ \ -\frac{1}{9}$	$\frac{1}{6},\ \ -\frac{1}{4}$	1	27
5	6	15	1	$\sqrt{\frac{8}{5}}$	$-\frac{1}{5}$	$\frac{2}{5},\ \ -\frac{4}{5}$	$\frac{2}{5},\ \ -\frac{4}{5}$	1	$\frac{1}{2}$
6	7	21	1	$\sqrt{15}$	$-\frac{1}{6}$	$\frac{9}{2},\ \ -6$	$1,\ \ -\frac{5}{2}$	1	$\frac{1}{81}$
22	23	253	1	$\sqrt{\frac{126}{11}}$	$-\frac{1}{22}$	$\frac{45}{22},\ \ -\frac{117}{44}$	$\frac{21}{44},\ \ -\frac{12}{11}$	1	$\frac{1}{81}$

For $n = 4,\ 5,\ 6$**,** X_2 **has the structure of the Johnson scheme** $J(n+1, 2)$**, that is, the trivial tight** 4**-design in** $J(n+1, 2)$**. For** $n = 22$**,** X_2 **has the structure of tight** 4**-**$(23, 7, 1)$ **design in the Johnson schem** $J(23, 7)$**.**

Theorem II

$|X_1| = n+2$,

n	$\lvert X_1\rvert$	$\lvert X_2\rvert$	r_1	r_2	$A(X_1)$	$A(X_2)$	$A(X_1, X_2)$	w_1	w_2
4	6	9	1	$\sqrt{2}$	$0,\ -\frac{1}{2}$	$\frac{1}{2},\ -1$	$\frac{1}{2},\ -1$	1	$\frac{1}{3}$

X_2 has the structure of the Hamming scheme $H(2,3)$, that is, trivial tight 4-design of the Hamming scheme.

Theorem III

n	$\lvert X_1\rvert$	$\lvert X_2\rvert$	r_1	r_2	$A(X_1)$	$A(X_2)$	$A(X_1, X_2)$	w_1	w_2
22	33	243	1	$\sqrt{11}$	$0,\ -\frac{1}{2}$	$2,\ -\frac{5}{2}$	$\frac{1}{2},\ -1$	1	$\frac{1}{81}$

X_2 has the structure of tight 4-design in the Hamming scheme $H(11,3)$.

Note that the inner products $A(Y_i)$ or $A(Y_i, Y_j)$ are differently normalized from the previous normalization in this talk.

For more details of this talk, see our paper:
Euclidean designs and coherent configurations
by Eiichi Bannai and Etsuko Bannai,
which will be available in arXiv:0905.2143.

THANK YOU

Appendix I

Feasible parameters of the Euclidean 4-design (X, w) given in Theorem C(2)(ii) and the intersection numbers of the coresponding coherent configuration.

$$
\begin{aligned}
& n = (2k-1)^2 - 4, \\
& |X_1| = 2(2k+1)(k-1)^3, \qquad |X_2| = 2k^3(2k-3), \\
& A(X_1, X_1) = \{\tfrac{k-2}{k(2k-3)},\ -\tfrac{1}{2k-3}\}, \qquad A(X_2, X_2) = \{\tfrac{1}{2k+1}, -\tfrac{k+1}{(k-1)(2k+1)}\}, \\
& A(X_1, X_2) = \{\tfrac{1}{\sqrt{n}},\ -\tfrac{1}{\sqrt{n}}\}, \\
& r_1 = 1, \qquad w_1 = 1, \qquad w_2 = \tfrac{(2k+1)^2(k-1)^4}{(2k-3)^2 k^4} r_2^{-4},
\end{aligned}
$$

Intersection matrices and Character tables of the association scheme for X_1

$$
B_1^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ k^3(2k-3) & (k+1)(k^2-k-1)k & (k-1)k^3 \\ 0 & (k^2-k-1)(k-1)^2 & k^3(k-2) \end{bmatrix},
$$

$$
B_2^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & (k^2-k-1)(k-1)^2 & k^3(k-2) \\ (k-1)(2k-3)(k^2-k-1) & (k-2)(k-1)(k^2-k-1) & (k-1)(k-2)(k^2-2k-2) \end{bmatrix},
$$

$$P_1 = \begin{bmatrix} 1 & k^3(2k-3) & (k-1)(2k-3)(k^2-k-1) \\ 1 & k^2(k-2) & -1-k^2(k-2) \\ 1 & -k & -1+k \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1 & (2k+1)(2k-3) & 2(2k-3)(k^2-k-1)k \\ 1 & \frac{(k-2)(2k+1)}{k} & -\frac{2(k^2-k-1)}{k} \\ 1 & -2k-1 & 2k \end{bmatrix},$$

Intersection matrices and Character tables of the association scheme for X_2

$$B_1^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ (2k+1)(k^2-k-1)k & (k+1)(k^2-3)k & (k+1)(k^2-k-1)k \\ 0 & (k+1)(k-1)^3 & (k^2-k-1)k^2 \end{bmatrix},$$

$$B_2^{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & (k+1)(k-1)^3 & (k^2-k-1)k^2 \\ (2k+1)(k-1)^3 & (k-1)^3k & (k-2)(k-1)(k^2-k-1) \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & (2k+1)(k^2-k-1)k & (k-1)(2k^3-3k^2+1) \\ 1 & k(k^2-k-1) & -(k-1)(k^2-1) \\ 1 & -k & k-1]]) \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & (2k+1)(2k-3) & 2(k-1)(2k+1)(k^2-k-1) \\ 1 & 2k-3 & -2k+2 \\ 1 & -\frac{(2k-3)(k+1)}{k-1} & \frac{2(k^2-k-1)}{k-1} \end{bmatrix}$$

$$p^{\alpha_0}_{\gamma_1,\gamma_1} = k^3(2k-3), \quad p^{\beta_0}_{\gamma_1,\gamma_1} = (2k+1)(k-1)^3$$

$$p^{\alpha_1}_{\gamma_2,\gamma_2} = (k^2-k-1)k^2, \quad p^{\alpha_1}_{\gamma_1,\gamma_2} = (k-1)^2k^2, \quad p^{\alpha_1}_{\gamma_1,\gamma_1} = (k^2-k-1)k^2$$

$$p^{\alpha_2}_{\gamma_2,\gamma_2} = k^3(k-2), \quad p^{\alpha_2}_{\gamma_1,\gamma_2} = (k-1)k^3, \quad p^{\alpha_2}_{\gamma_1,\gamma_1} = k^3(k-2)$$

$$p^{\beta_1}_{\gamma_1,\gamma_2} = (k-1)^3 k, \quad p^{\beta_1}_{\gamma_2,\gamma_2} = (k+1)(k-1)^3, \quad p^{\beta_1}_{\gamma_1,\gamma_1} = (k+1)(k-1)^3$$

$$p^{\beta_2}_{\gamma_1,\gamma_2} = (k-1)^2 k^2, \quad p^{\beta_2}_{\gamma_2,\gamma_2} = (k^2-k-1)(k-1)^2, \quad p^{\beta_2}_{\gamma_1,\gamma_1} = (k^2-k-1)(k-1)^2,$$

$$p^{\gamma_1}_{\gamma_2,\beta_2} = (k-1)^2 k^2, \quad p^{\gamma_1}_{\alpha_2,\gamma_2} = (k^2-k-1)(k-1)^2, \quad p^{\gamma_1}_{\gamma_1,\beta_1} = (k+1)(k^2-k-1)k,$$

$$p^{\gamma_1}_{\gamma_2,\beta_1} = (k^2-k-1)k^2 \ , \quad p^{\gamma_1}_{\alpha_1,\gamma_1} = (k^2-k-1)k^2, \quad p^{\gamma_1}_{\alpha_1,\gamma_2} = (k-1)^2 k^2,$$

$$p^{\gamma_1}_{\gamma_1,\beta_2} = (k^2-k-1)(k-1)^2, \quad p^{\gamma_1}_{\alpha_2,\gamma_1} = (k-2)(k-1)(k^2-k-1),$$

$$p^{\gamma_2}_{\gamma_2,\beta_2} = (k^2-k-1)(k-1)^2 \ , \quad p^{\gamma_2}_{\alpha_2,\gamma_2} = (k-2)(k-1)(k^2-k-1),$$

$$p^{\gamma_2}_{\gamma_1,\beta_2} = (k-1)^2 k^2, \quad p^{\gamma_2}_{\gamma_1,\beta_1} = (k^2-k-1)k^2, \quad p^{\gamma_2}_{\alpha_1,\gamma_2} = (k^2-k-1)k^2,$$

$$p^{\gamma_2}_{\alpha_2,\gamma_1} = (k^2-k-1)(k-1)^2, \quad p^{\gamma_2}_{\gamma_2,\beta_1} = (k+1)(k^2-k-1)k, \quad p^{\gamma_2}_{\alpha_1,\gamma_1} = (k-1)^2 k^2.$$

In above $p^c_{a,b} = p^c_{b,a}$ holds for any $a, b, c \in \{\alpha_i, \beta_j, \gamma_k \mid i,\ j = 0, 1, 2, k = 1, 2\}$.

Appendix II

The feasible parameters of the Euclidean tight 4-design given in Theorem E and intersection numbers of the corresponding coherent configuration.

$$n = (6k-3)^2 - 3,$$

$$|X_1| = (6k^2-6k+1)(36k^2-36k+7), \quad |X_2| = 3(36k^2-36k+7)(2k-1)^2,$$

$$A(X_1,X_1) = \left\{ \frac{18k^2-27k+8}{6(9k^2-9k+1)(2k-1)}, \quad -\frac{18k^2-9k-1}{6(9k^2-9k+1)(2k-1)} \right\},$$

$$A(X_2,X_2) = \left\{ \frac{36k^3-54k^2+25k-4}{2(6k^2-6k+1)(18k^2-18k+5)}, \quad -\frac{36k^3-54k^2+25k-3}{2(6k^2-6k+1)(18k^2-18k+5)} \right\},$$

$$A(X_1,X_2) = \left\{ \sqrt{\frac{36k^2-36k+4}{(36k^2-36k+6)(36k^2-36k+10)}}, -\sqrt{\frac{36k^2-36k+10}{(36k^2-36k+6)(36k^2-36k+4)}} \right\},$$

$$r_1 = 1, \quad r_2 = \sqrt{\frac{3(18k^2-18k+5)(6k^2-6k+1)}{9k^2-9k+1}},$$

$$w_1 = 1, \quad w_2 = \frac{1}{81(2k-1)^4}.$$

Intersection matrices and the Character tables of the association scheme for X_1

$$B_1^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 6(-1+2k)(9k^2-9k+1)k & 54k^4-45k^3-12k^2+7k+1 & (18k^2-9k-1)k(3k-2) \\ 0 & (3k-2)(k-1)(18k^2-9k-1) & k(3k-1)(18k^2-27k+8) \end{bmatrix},$$

$$B_1^{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & (3k-2)(k-1)(18k^2-9k-1) & k(3k-1)(18k^2-27k+8) \\ 6(k-1)(-1+2k)(9k^2-9k+1) & (18k^2-27k+8)(k-1)(3k-1) & 54k^4-171k^3+177k^2-64k+5 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 1 & 6(-1+2k)(9k^2-9k+1)k & 6(k-1)(-1+2k)(9k^2-9k+1) \\ 1 & -3k+1 & 3k-2 \\ 1 & k(18k^2-27k+8) & -(k-1)(18k^2-9k-1) \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1 & 6(36k^2-36k+7)(k-1)k & 36k^2-36k+6 \\ 1 & -\frac{(3k-1)(k-1)(36k^2-36k+7)}{(-1+2k)(9k^2-9k+1)} & \frac{(18k^2-27k+8)(6k^2-6k+1)}{(-1+2k)(9k^2-9k+1)} \\ 1 & \frac{k(3k-2)(36k^2-36k+7)}{(-1+2k)(9k^2-9k+1)} & -\frac{(18k^2-9k-1)(6k^2-6k+1)}{(-1+2k)(9k^2-9k+1)} \end{bmatrix},$$

Intersection matrices and the Character tables of the association scheme for X_2

$$B_2^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 2(6k^2-6k+1)(18k^2-18k+5) & (9k^2-9k+1)(12k^2-10k+3) & (3k-2)(36k^3-54k^2+25k-3) \\ 0 & (3k-2)(36k^3-54k^2+25k-3) & (36k^3-54k^2+25k-4)(3k-1) \end{bmatrix},$$

$$B_2^{(2)} = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & (3k-2)(36k^3-54k^2+25k-3) & (36k^3-54k^2+25k-4)(3k-1) \\ 2(6k^2-6k+1)(18k^2-18k+5) & (36k^3-54k^2+25k-4)(3k-1) & (9k^2-9k+1)(12k^2-14k+5) \end{array} \right],$$

$$P_2 = \left[\begin{array}{ccc} 1 & 2(6k^2-6k+1)(18k^2-18k+5) & 2(6k^2-6k+1)(18k^2-18k+5) \\ 1 & -3k+1 & 3k-2 \\ 1 & 36k^3-54k^2+25k-4 & 3-36k^3+54k^2-25k \end{array} \right],$$

$$Q_2 = \left[\begin{array}{ccc} 1 & 2(6k^2-6k+1)(36k^2-36k+7) & 36k^2-36k+6 \\ 1 & -\frac{(3k-1)(36k^2-36k+7)}{18k^2-18k+5} & \frac{3(36k^3-54k^2+25k-4)}{18k^2-18k+5} \\ 1 & \frac{(3k-2)(36k^2-36k+7)}{18k^2-18k+5} & -\frac{3(36k^3-54k^2+25k-3)}{18k^2-18k+5} \end{array} \right],$$

$$p^{\alpha_0}_{\gamma_1,\gamma_1} = 3(18k^2-18k+5)(2k-1)^2, \quad p^{\beta_0}_{\gamma_1,\gamma_1} = (6k^2-6k+1)(18k^2-18k+5),$$

$$p^{\alpha_1}_{\gamma_2,\gamma_2} = (2k-1)(54k^3-72k^2+15k+4), \quad p^{\alpha_1}_{\gamma_1,\gamma_2} = (3k-2)(2k-1)(18k^2-18k+5),$$
$$p^{\alpha_1}_{\gamma_1,\gamma_1} = (2k-1)(3k-1)(18k^2-18k+5),$$

$$p^{\alpha_2}_{\gamma_2,\gamma_2} = (54k^3-90k^2+33k-1)(2k-1), \quad p^{\alpha_2}_{\gamma_1,\gamma_2} = (2k-1)(3k-1)(18k^2-18k+5),$$
$$p^{\alpha_2}_{\gamma_1,\gamma_1} = (3k-2)(2k-1)(18k^2-18k+5),$$

$$p^{\beta_1}_{\gamma_1,\gamma_2} = (2k-1)(3k-2)(9k^2-9k+1), \quad p^{\beta_1}_{\gamma_2,\gamma_2} = (9k^2-9k+1)k(6k-5),$$
$$p^{\beta_1}_{\gamma_1,\gamma_1} = (3k-1)(18k^3-27k^2+14k-3),$$

$$p^{\beta_2}_{\gamma_1,\gamma_2} = (3k-1)(9k^2-9k+1)(2k-1), \quad p^{\beta_2}_{\gamma_2,\gamma_2} = (9k^2-9k+1)(6k-1)(k-1),$$

$$
\begin{aligned}
p^{\beta_2}_{\gamma_1,\gamma_1} &= (3k-2)(18k^3-27k^2+14k-2),
\end{aligned}
$$

$$
\begin{aligned}
p^{\gamma_1}_{\gamma_2,\beta_2} &= 2(3k-1)(9k^2-9k+1)(2k-1), & p^{\gamma_1}_{\alpha_2,\gamma_2} &= 2(3k-1)(k-1)(9k^2-9k+1),\\
p^{\gamma_1}_{\gamma_1,\beta_1} &= 2(3k-1)(18k^3-27k^2+14k-3), & p^{\gamma_1}_{\gamma_2,\beta_1} &= 2(2k-1)(3k-2)(9k^2-9k+1),\\
p^{\gamma_1}_{\alpha_1,\gamma_1} &= 2k(3k-1)(9k^2-9k+1), & p^{\gamma_1}_{\alpha_1,\gamma_2} &= 2k(3k-2)(9k^2-9k+1),\\
p^{\gamma_1}_{\gamma_1,\beta_2} &= 2(3k-2)(18k^3-27k^2+14k-2), & p^{\gamma_1}_{\alpha_2,\gamma_1} &= 2(k-1)(9k^2-9k+1)(3k-2),
\end{aligned}
$$

$$
\begin{aligned}
p^{\gamma_2}_{\gamma_2,\beta_2} &= (6k-1)(k-1)(18k^2-18k+5), & p^{\gamma_2}_{\alpha_2,\gamma_2} &= (k-1)(54k^3-90k^2+33k-1),\\
p^{\gamma_2}_{\gamma_1,\beta_2} &= (2k-1)(3k-1)(18k^2-18k+5), & p^{\gamma_2}_{\gamma_1,\beta_1} &= (3k-2)(2k-1)(18k^2-18k+5),\\
p^{\gamma_2}_{\alpha_1,\gamma_2} &= k(54k^3-72k^2+15k+4), & p^{\gamma_2}_{\alpha_2,\gamma_1} &= (18k^2-18k+5)(3k-1)(k-1),\\
p^{\gamma_2}_{\gamma_2,\beta_1} &= (6k-5)k(18k^2-18k+5), & p^{\gamma_2}_{\alpha_1,\gamma_1} &= (3k-2)k(18k^2-18k+5).
\end{aligned}
$$

In above $p^c_{a,b} = p^c_{b,a}$ **holds for any** $a, b, c \in \{\alpha_i, \beta_j, \gamma_k \mid i,\ j = 0, 1, 2, k = 1, 2\}$**.**