k-Parabolic Subspace Arrangements

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Fadell, Fox, Neuwirth, 1963

Take an n dimensional complex space, delete all diagonals $z_i = z_j$

$\mathbb{C}^n - D$ is a $K(\pi, 1)$ space, with fundamental group $\cong$ pure braid group

Khovanov, 1996 **(real counterpart)**

Take an n dimensional real space, delete all co-dimension 2 subspaces, $x_i = x_j = x_k$

$\mathbb{R}^n - X_n$ is a $K(\pi, 1)$ space, with fundamental group $\cong$ pure twin group

Brieskorn, Deligne 1970's

Consider $\mathcal{H}$, the complexification of a Coxeter arrangement of type W

$\mathbb{C}^n - \mathcal{H}$ is a $K(\pi, 1)$ space, with fundamental group $\cong$ pure Artin group, of type W.

B. Severs, White, 2008 **(real counterpart)**

Take an n dimensional real space, delete $\mathcal{P}_W$, the set of all 3-parabolic subspaces of type W

$\mathbb{R}^n - \mathcal{P}_W$ is a $K(\pi, 1)$ space (**claim**) with

fundamental group $\cong$ pure triplet group of type W **(Theorem)**

Discrete Homotopy Theory

W - an irreducible finite real reflection group acting on $\mathbb{R}^n$, with:

- $S \subset W$ a set of simple reflections,
- $R = \{wsw^{-1} : s \in S, w \in W\}$ the set of all reflections.
- $m(s,s) = 1, m(s,t) = m(t,s)$ for all $s,t \in S$

and

W is generated by S subject to:

1. $s^2 = 1, \forall s \in S$
2. $st = ts, \forall s, t \in S$ such that $m(s,t) = 2$
3. $sts = tst, \forall s, t \in S$, such that $m(s,t) = 3$

$$\vdots$$

i. $\underbrace{stst\cdots}_{i} = \underbrace{tsts\cdots}_{i}, \forall s, t \in S$, such that $m(s,t) = i$

$$\vdots$$

Definition

The Coxeter arrangement $\mathcal{H}(W)$ is given by hyperplanes

$$H_r = \{x \in \mathbb{R}^n : rx = x\}$$

for each $r \in R$.

Example: Braid Arrangement

When W is of type A, the Coxeter arrangement is given by

$$x_i - x_j = 0, 1 \leq i < j \leq n+1$$

and

$\pi_1(\mathbb{C}^n - \mathcal{H}_{\mathcal{A}}) \cong$ pure braid group
$\pi_1(\mathbb{R}^n - \mathcal{H}_{\mathcal{W}}) \cong$ pure Artin group of type W.

What is an Artin group of type W over $\mathbb{C}$

W^1 is generated by $S \in W$ subject to:

1. ~~$s^2 = 1, \forall s \in S$~~
2. $st = ts$, $\forall s, t \in S$ such that $m(s,t) = 2$
3. $sts = tst$, $\forall s, t \in S$, such that $m(s,t) = 3$

$$\vdots$$

i. $\underbrace{stst\cdots}_{i} = \underbrace{tsts\cdots}_{i}$, $\forall s, t \in S$, such that $m(s,t) = i$

$$\vdots$$

Let

$\varphi : W^1 \rightarrow W$ with $\varphi(s) = s$, for all $s \in S$.

$\ker \varphi$ = pure Artin Group of type W.

Brieskorns 1973

$\mathcal{H}_{\mathcal{W}}$ a Coxeter arrangement of type W

$$\pi_1(\mathbb{C}^n - \mathcal{H}_{\mathcal{W}}) \cong \ker \varphi$$

Definition

The k-equal arrangment, $\mathcal{A}_{n,k}$ consists of subspaces (of $\mathbb{R}^n$) given by equations:

- $x_{i_1} = x_{i_2} = \ldots = x_{i_k}$,
  for all distinct indices $1 \leq i_1 < \ldots < i_k \leq n$

- When k=2 we recover the Braid arrangement.

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  for all distinct indices $1 \leq i_1 < \ldots < i_k \leq n$

- When k=2 we recover the Braid arrangement.
- Khovanov (1996) gave a description of $\pi_1(\mathcal{M}(\mathcal{A}_{n,3}))$ as a Pure Twin Group.
- He also showed that $\mathcal{M}^{\mathbb{R}}(\mathcal{A}_{n,3})$ is a $K(\pi, 1)$ space.

What is a twin group over $\mathbb{R}$

W^3 is generated by $S \in S_n$ subject to:

1. $s^2 = 1, \forall s \in S$
2. $st = ts, \forall s, t \in S$ such that $m(s,t) = 2$
3. ~~$sts = tst, \forall s, t \in S$, such that $m(s,t) = 3$~~

Pure twin group

Let

$\varphi : W^3 \to W$ with $\varphi(s) = s$, for all $s \in S$.

ker φ = pure Twin Group $\cong \pi_1(\mathcal{M}(\mathcal{A}_{n,3}))$

Real analogue to Brieskorn results in the spirit of Khovanov

W^i is generated by $S \subset W$ subject to:

1. $s^2 = 1, \forall s \in S$
2. $st = ts, \forall s, t \in S$ such that $m(s,t) = 2$
3. ~~$sts = tst, \forall s, t \in S$, such that $m(s,t) = 3$~~

$$\vdots$$

i. ~~$\underbrace{stst\cdots}_{i} = \underbrace{tsts\cdots}_{i}, \forall s, t \in S$, such that $m(s,t) = i$~~

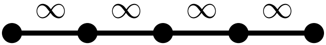
$$\vdots$$

Example with Dynkin Diagrams

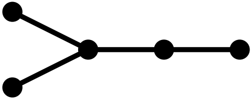
Comparing Dynkin diagrams of W and W':

A_5

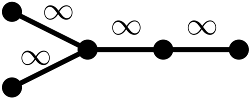
A'_5



D_5



D'_5



Let $\varphi : W^i \to W$ given by $\varphi(s) = s$, for all $s \in S$.

Theorem

B., Severs, White, (2008)

$$\ker \varphi \cong \pi_1(\mathcal{M}^{\mathbb{R}}(\mathcal{W}_{n,3})),$$

where $\mathcal{W}_{n,k}$ is the k-parabolic arrangement of type w.

Definition

- A subgroup $G \subseteq W$ is parabolic if $G = < wIw^{-1} >$, for some $I \subset S$, $w \in W$.
  G is k-parabolic if G is of rank $k - 1$.

Definition

- A subgroup $G \subseteq W$ is parabolic if $G =< wIw^{-1} >$,
  for some $I \subset S$, $w \in W$.
  G is k-parabolic if G is of rank $k - 1$.
- For $G \subset W$, let $Fix(G) = \{x \in \mathbb{R}^n : wx = x, \forall w \in G\}$
- For subspace $X \subset \mathbb{R}^n$, let $Gal(X) = \{w \in W : wx = x, \forall x \in X\}$

Let $\mathcal{P}(W)$ be the poset of all parabolic subgroups of W ordered by inclusion. Let $\mathcal{L}(W)$ be the intersection lattice of the Coxeter arrangement, ordered by reverse inclusion.

Theorem (Barcelo and Ihrig, 1999)

$\mathcal{P}(W) \cong \mathcal{L}(W)$ *via*

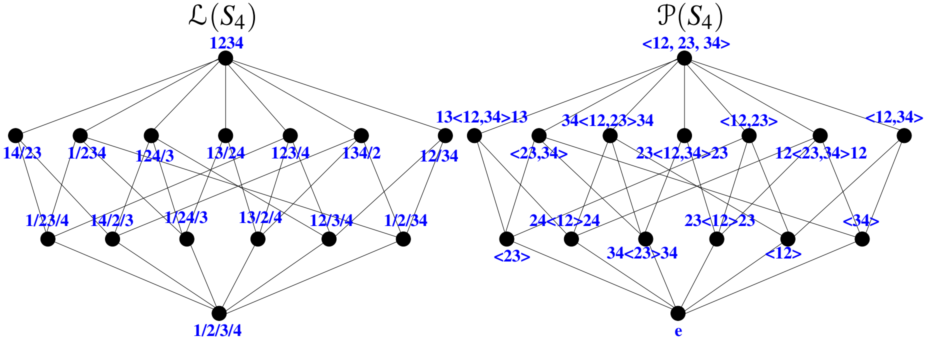
$$G \longrightarrow Fix(G)$$

$$Gal(X) \longleftarrow X$$

We will use this "Galois correspondence" to define k-parabolic arrangements.

But first we give an example for $A_3 = S_4$.

Example of correspondence $W = S_4$



Example: $14/23 \leftrightarrow <(1,4),(2,3)> = (1,3) < (1,2),(3,4) > (1,3)$

Example: $134/2 \leftrightarrow <(1,3),(3,4)> = (1,2) < (2,3),(3,4) > (1,2)$

Definition

Let W be an irreducible real reflection group of rank n. Let $\mathcal{P}_{n,k}(W)$ contain all irreducible k-parabolic subgroups of W.
Then the k-parabolic arrangement $\mathcal{W}_{n,k}$ is the collection of subspaces

$$Fix(G), G \in \mathcal{P}_{n,k}(W)$$

Example: $W = S_9 = A_8$

Let $G = (1,4)(6,8) < (4,5),(5,6) > (6,8)(1,4) =< (1,4),(4,8) >$.

- We see that $Fix(G)$ is given by $x_1 = x_4 = x_8$.
- For every $G \in \mathcal{P}_{8,3}(A_8)$, $Fix(G)$ is a subspace in $\mathcal{A}_{9,3}$.
- Thus, $\mathcal{W}_{8,3}$ is the 3-equal arrangement in $\mathbb{R}^8$.

Examples of the k-parabolic arrangement

- When W is of type A or B, then $\mathcal{W}_{n,k}$ corresponds to $\mathcal{A}_{n+1,k}$, and the $\mathcal{B}_{n,k,k-1}$ arrangement (of Björner-Welker and Björner-Sagan respectively).
- $\mathcal{W}_{n,2}$ is the Coxeter arrangement for W and $\mathcal{W}_{n,n+1}$ consists of the origin.
- When W is of type D, then $\mathcal{W}_{n,3}$ corresponds to the Björner-Sagan $\mathcal{D}_{n,3}$ arrangement (not so for $\mathcal{W}_{n,k}$, $k > 3$).

Why is B., Severs and White's Theorem true?

Essentially because

$$\pi_1(\mathcal{M}^{\mathbb{R}}(\mathcal{W}_{n,3})) \cong A_1^{n-2}(W - permutahedron)$$

and

$$\ker(\varphi) \cong A_1^{n-2}(W - permutahedron).$$

Definition

Let Δ be simplicial complex of dimension d, $0 \leq q \leq d$, $\sigma_0 \in \Delta$ be maximal with dimension $\geq q$.

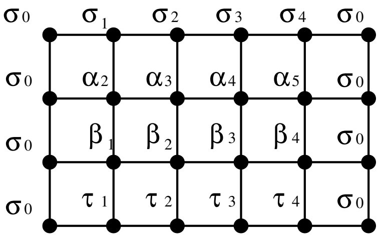
1. Two simplicies σ and τ are q-near if $|\sigma \cap \tau| \geq q + 1$.
2. A q-chain is a sequence $\sigma_1, \ldots, \sigma_k$, such that σ_i, σ_{i+1} are q-near for all i.
3. A q-loop is a q-chain with $\sigma_1 = \sigma_k = \sigma_0$.

Definition

We define an equivalence relation, $\simeq_A$ on q-loops with the following conditions:

1. The q-loop
$$(\sigma) = (\sigma_0, \sigma_1, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n, \sigma_0)$$
is equivalent to the q-loop
$$(\sigma) = (\sigma_0, \sigma_1, \ldots, \sigma_i, \sigma_i, \sigma_{i+1}, \ldots \sigma_n, \sigma_0)$$
2. If (σ) and (τ) have the same length then they are equivalent if there is a grid between them.

A-Homotopy



Edges between two simplices indicate they are q-near. Each row is a q-loop.

Such a grid is an A-homotopy between (σ) and(τ).

The equivalence relation $\simeq_A$ is called A-homotopy.

The set of equivalence classes, $A_1^q(\Delta, \sigma_0)$, forms a group under concatenation.

Definition

Let $\Gamma = \Gamma^q(\Delta)$ be a graph with the following properties:

1. The vertices of Γ are the maximal simplices of Δ.
2. $\sigma\tau$ is an edge iff they are q-near.

Theorem (B., Kramer, Laubenbacher, Weaver, 2001)

$$A_1^q(\Delta, \sigma_0) \simeq \pi_1(X_\Gamma, \sigma_0)$$

where X_Γ *is a cell complex obtained by gluing a 2-cell on each 3- and 4-cycle of* $\Gamma = \Gamma^q(\Delta)$.

Given W with essentialized Coxeter arrangement $\mathcal{H}$, intersect the Coxeter arrangement with the $(n-1)$-sphere.

The resulting cell decomposition of the sphere is the Coxeter complex, $\mathcal{C}(W)$.

Theorem (B., Severs, White (2008))

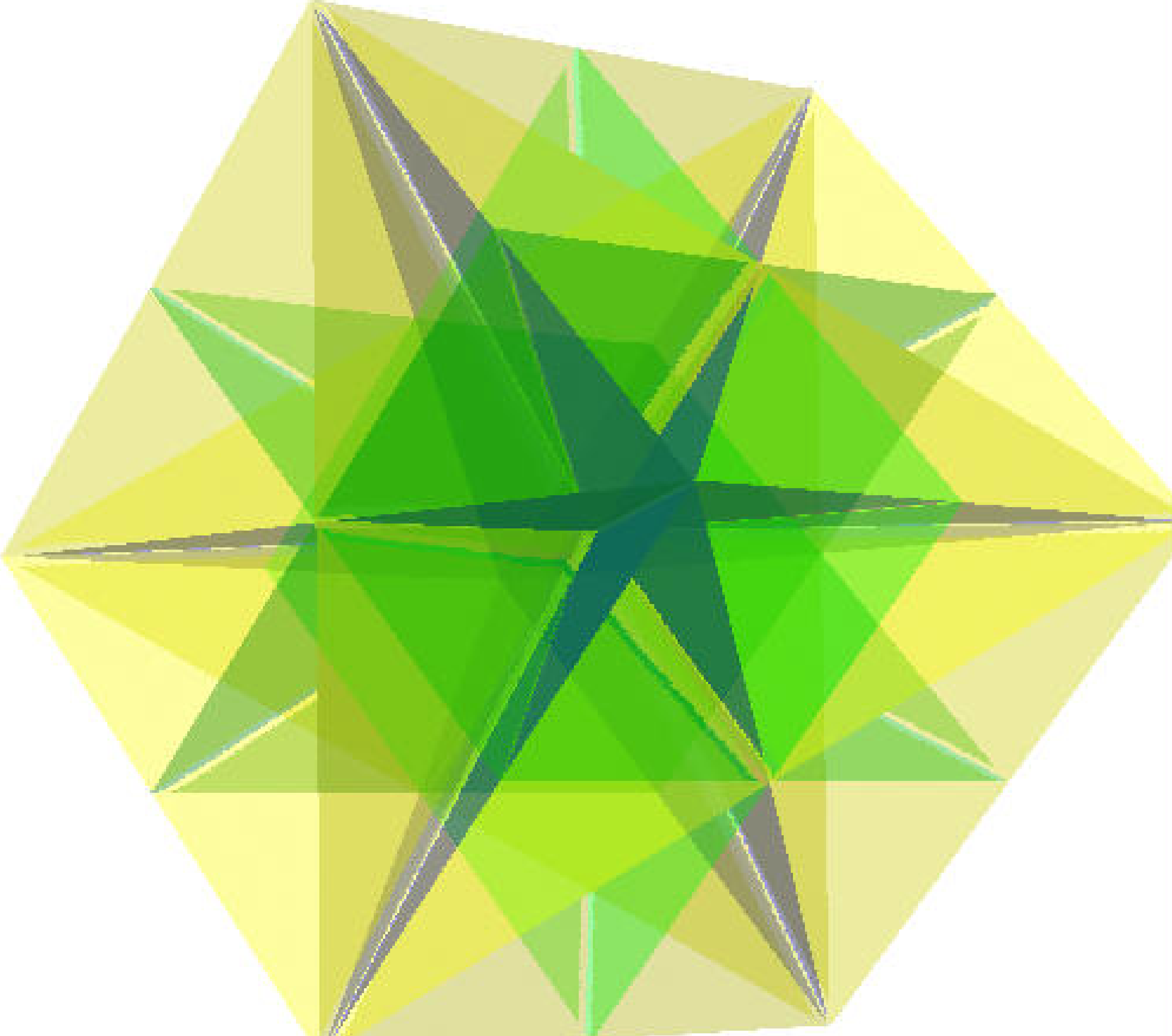
$$\pi_1(\mathcal{M}(\mathcal{W}_{n,3})) \cong A_1^{n-2}(\mathcal{C}(W)).$$

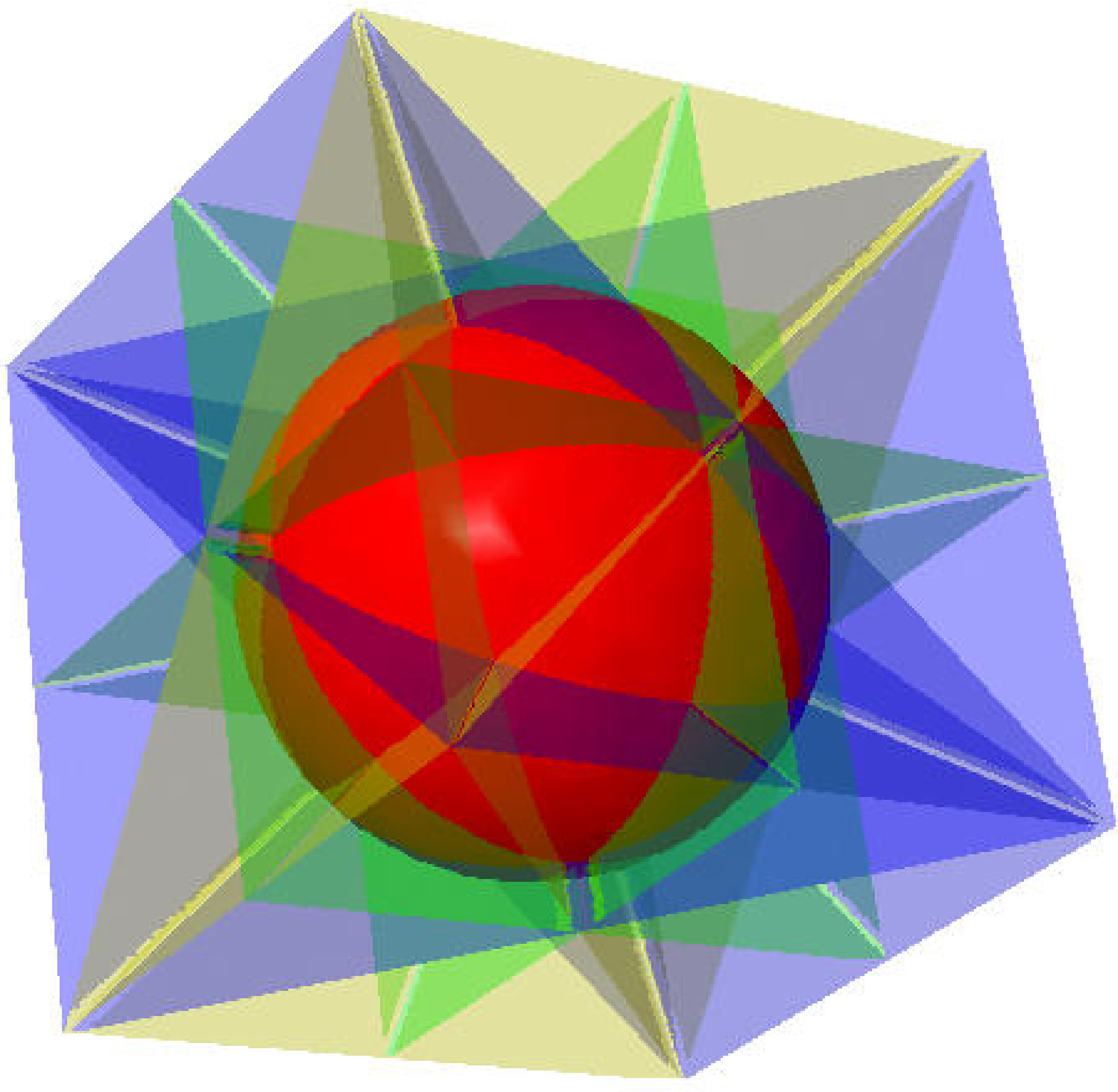
Why is $A_1^{n-2}(\mathcal{C}(W)) \cong \ker \varphi$

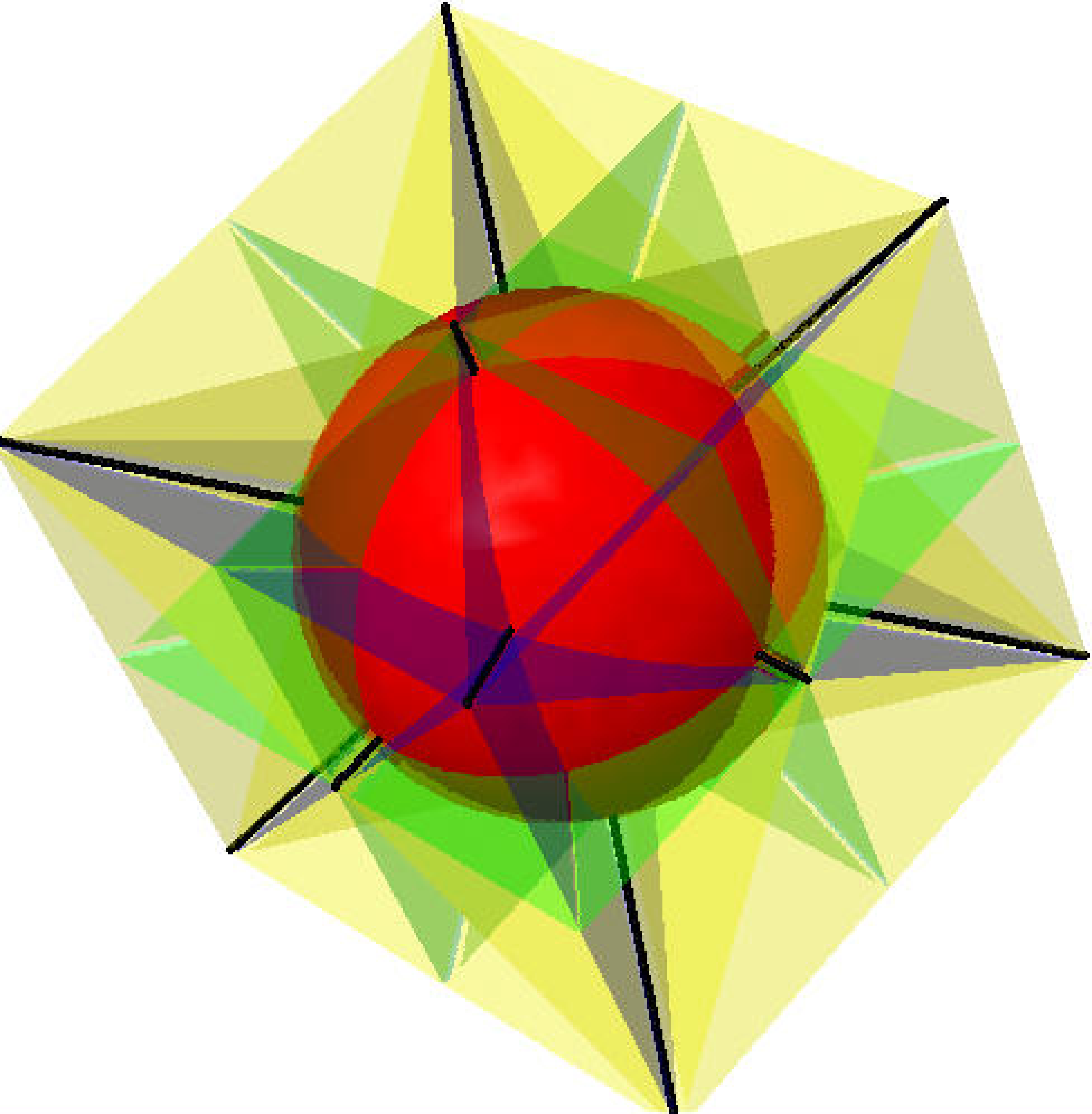
- $\Gamma = \Gamma^{n-2}(\mathcal{C}(W))$ is the graph of the W-Permutahedron.
- Vertices in Γ correspond to elements of W.
- σ, τ is an edge if $\sigma = \tau s$ for some $s \in S$. Label the edge σ, τ by s.
- Γ is bipartite, labels of 4-cycles correspond to pairs s, t of commuting reflections.

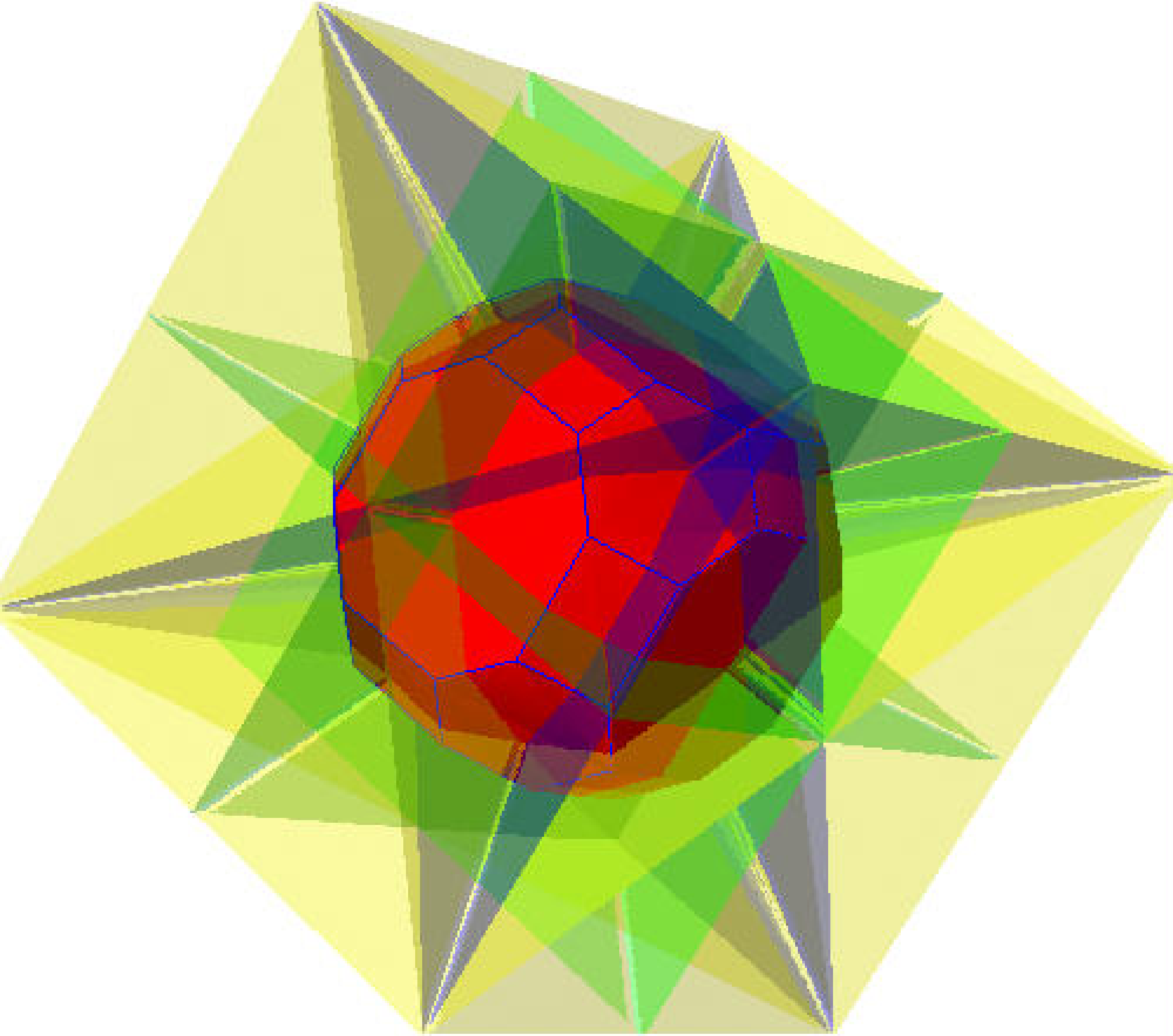
- Walks in $\Gamma^{n-2}(\mathcal{C}(W)) \leftrightarrow$ words in S^*
- Loops in $\Gamma^{n-2}(\mathcal{C}(W)) \leftrightarrow w \in S^*, w = 1 \in W$
- homotopic loops in $\Gamma^{n-2}(\mathcal{C}(W)) \leftrightarrow w = v \in \ker \varphi$

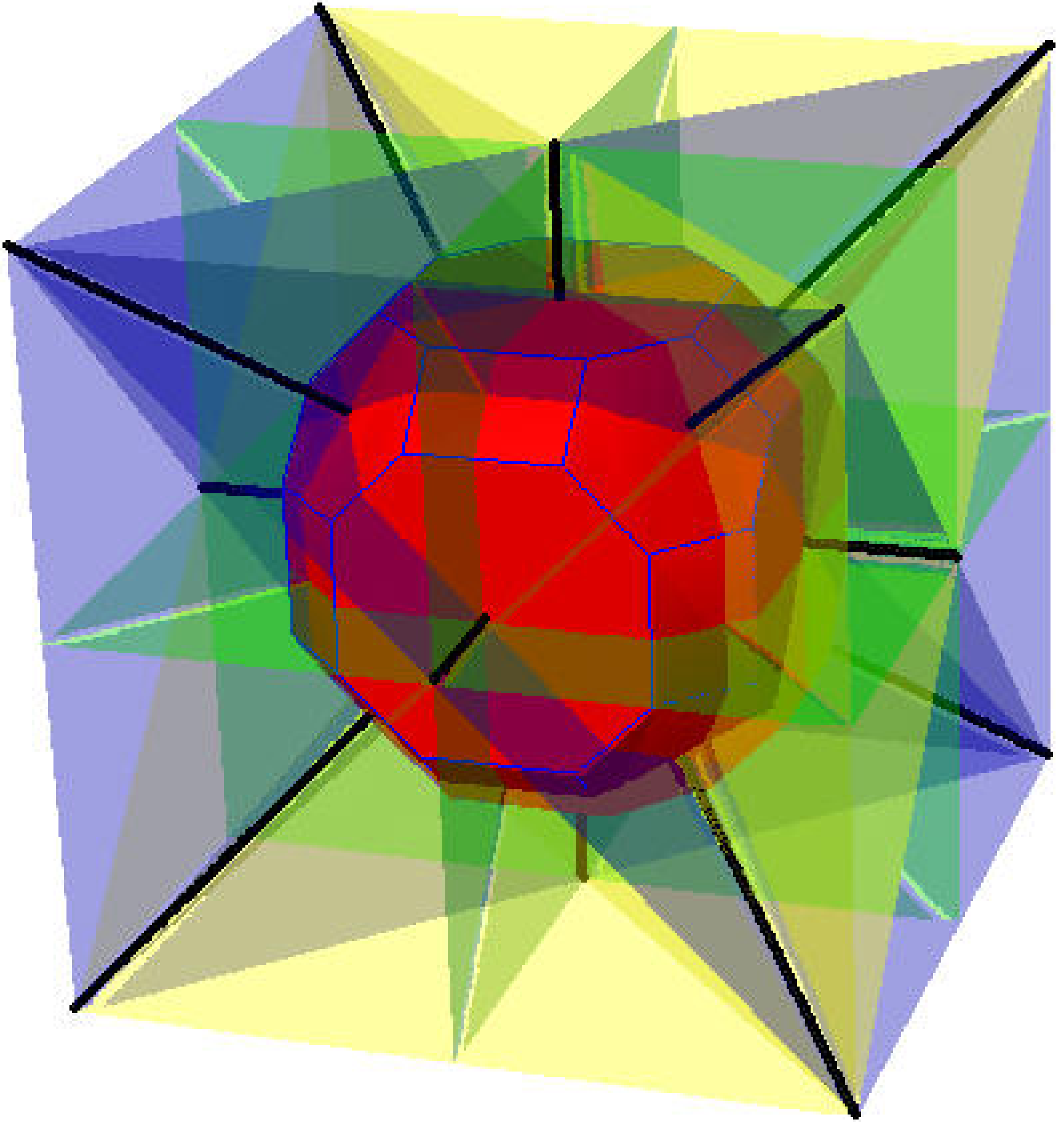
Thus one obtains that $\pi_1(\mathcal{M}(\mathcal{W}_{n,3})) \cong \ker \varphi$.

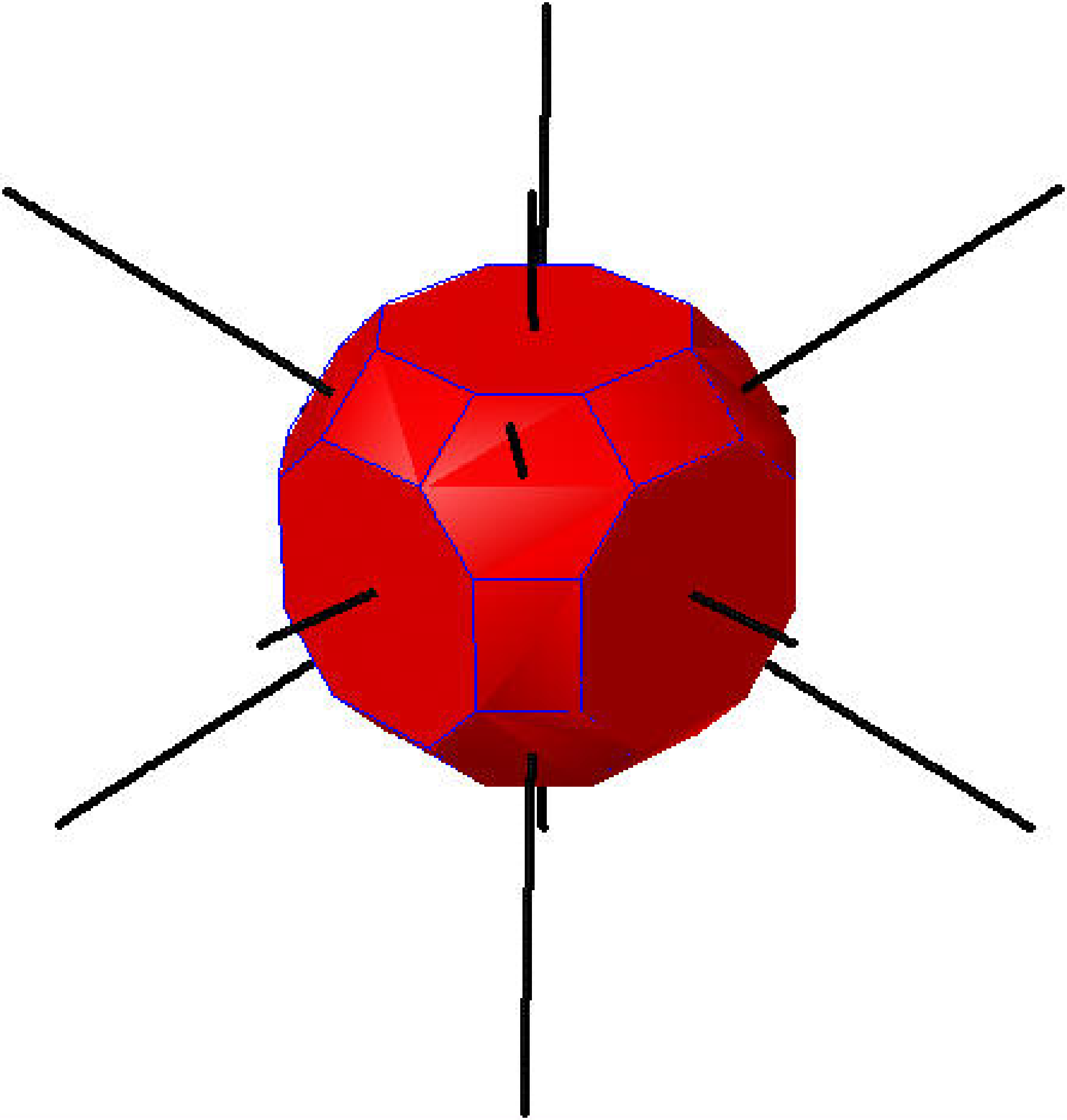


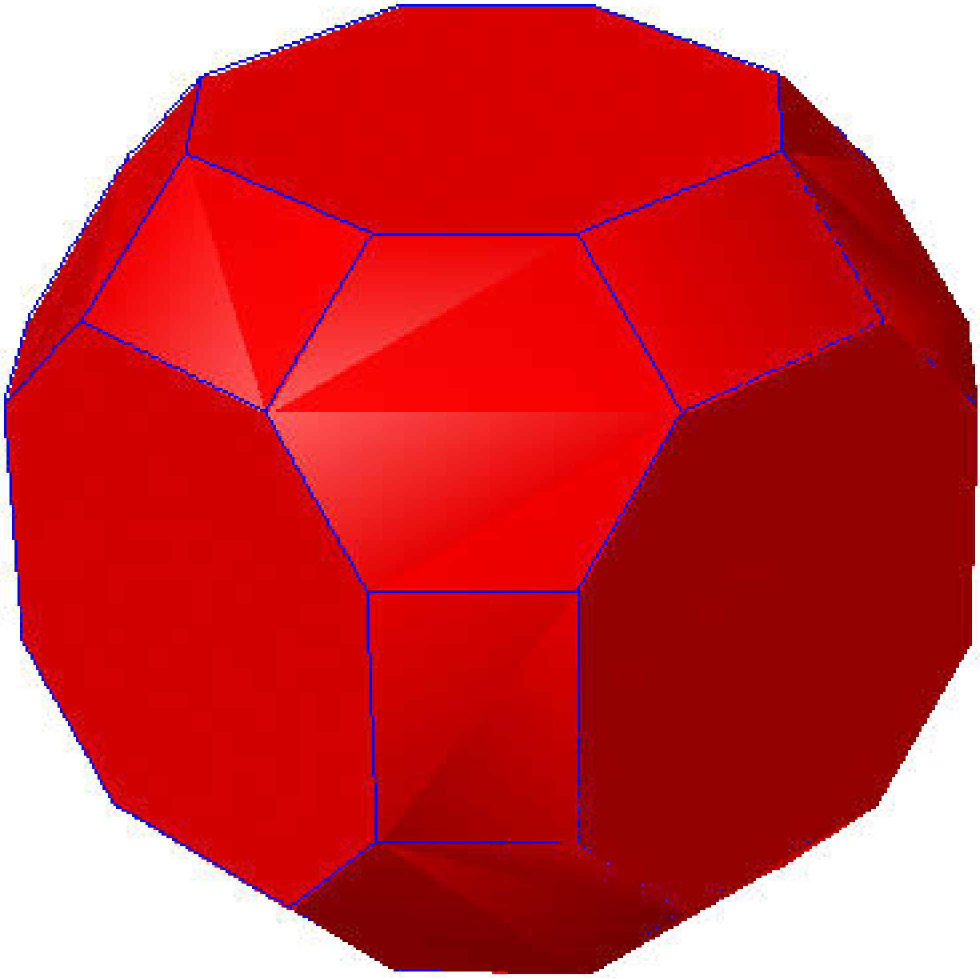












- Is homotopy theory of $\mathcal{M}(\mathcal{W}_{n,k})$ equivalent to A-theory of $\mathcal{C}(W)$? In other words, do we have:

$$A_m^{n-k}(\mathcal{C}(W)) \cong \pi_m(\mathcal{M}(\mathcal{W}_{n,k})), m \geq 1?$$

- If so, can we use combinatorial methods to calculate the rank of $H^{k-1}(\mathcal{M}(\mathcal{W}_{n,k}))$?
- Can the Betti numbers be formulated in terms of (combinatorial) invariants of Coxeter groups?
- Can we find a discrete homology theory?

Any Questions?

Thank You.