# Signed Domination Number of a Graph/Matrix

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# Dominating Signing of a Graph

$G = (V, E)$ a graph. For $v \in V$, $N[v]$ denotes the closed neighborhood of $v$.

A function $f : V \to \{1, -1\}$ is a **dominating (vertex)signing** of $G$ provided $\sum_{u \in N[v]} f(u) \geq 1$ for all vertices $v$.

The **value** $\gamma_{sd}(f)$ of a dominating signing $f$ is

$$\sum_{v \in V} f(v),$$

and the **signed domination number** of $G$ is

$$\gamma_{sd}(G) = \min\{\gamma_{sd}(f) : f \text{ a dominating signing of } G\}.$$

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# Dominating Edge Signing of a Graph

## Dominating Signing of a Graph

There are several lower and upper bounds known for $\gamma_{sd}(G)$ of order $n$ and exact values for certain graphs.

For instance (Haas and Wexler, 2001), if the minimum degree of $G$ is at least 2, then

$$\gamma_{sd}(G) \geq \frac{4-\Delta}{4+\Delta} n.$$

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# Dominating Edge Signing Number of a Graph

Let $L(G)$ be the **line graph** of $G = (V, E)$. The set of vertices of $L(G)$ is $E$, and for distinct $e, f \in E$, $e$ and $f$ are joined by an edge in $L(G)$ if and only if $e$ and $f$ have a common vertex.

A dominating signing of $L(G)$ is an **edge dominating signing** of $G$, so a function $h : E \to \{1, -1\}$ satisfying

$$\sum_{f \in N[e]} h(f) \geq 1 \text{ for all } e \in E.$$

Here $N[e]$ is the closed neighborhood of the edge $e$ in $G$.

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The **value** $\gamma'_{sd}(h)$ of an edge dominating signing $h$ is

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The **signed edge domination number** of $G$ is

$$\gamma'_{sd}(G) = \min_h \gamma'_{sd}(h) = \min_h \left\{ \sum_{e \in E} h(e) \right\}$$

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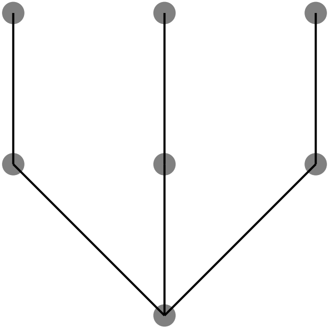
## Example

A tree $T$ with 7 vertices, 6 edges:



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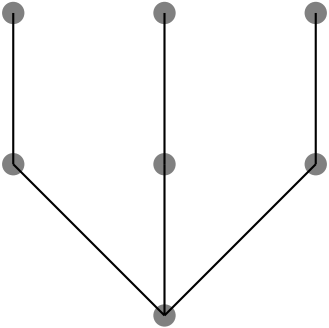

$$\gamma'_{sd}(T) = 6, \quad \text{all edges must be signed } +1$$

## Some Known Results on $\gamma'_{sd}(G)$

1. (Xu 2004) and (Karami, Khodkar, Sheikholeslami (2004/5) If $T$ is a tree, then $\gamma'_{sd}(T) \geq 1$.
2. (Xu 2005) More generally, if $G$ is a graph with $n$ vertices and $m$ edges, and no vertices of degree 0, then

$$n - m \leq \gamma'_{sd}(G) \leq 2n - 4.$$

   For every graph $G$ the graph $H$ obtained from $G$ by adding enough pendent edges to each vertex (one less that the degree of each vertex) achieves the lower bound.
3. (Karami, Sheikholeslami, Khodkar 2008) If $G$ is connected, then $\gamma'_{sd}(G) = n - m$ iff the degree of each vertex $v_i$ of $G$ is an odd number $2k_i + 1$ and there are at least $k_i$ pendent edges at $v_i$.

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## More Known Results on $\gamma'_{sd}(G)$

1. (Xu 2005) **Conjecture:** If $G$ is a graph with $n$ vertices, then
$$\gamma'_{sd}(G) \leq n - 1,$$
independent of the number $m$ of edges.
2. (Karami, Sheikholeslami, Khodkar 2008) The conjecture is true for Eulerian graphs, graphs with all vertices of odd degree, and regular graphs. In addition,
$$\gamma'_{sd}(G) \leq \lceil 3n/2 \rceil.$$
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# Bipartite Graphs

Let $G \subset K_{m,n}$ be a **bipartite graph** with bipartition $\{w_1, w_2, \dots, w_m\}$ and $\{b_1, b_2, \dots, b_n\}$.

WLOG, $G$ can be replaced with its $m$ by $n$ $(0,1)$-**biadjacency matrix** $A = [a_{ij}]$, where the **edges** of $G$ correspond to the **1s** in $A$.

A **signing** $h$ **of the edges of** $G$ corresponds to a **signing** $h$ **(of the 1s) of** $A$, resulting in a $(0, \pm 1)$-matrix $A'$:

$$h : A \rightarrow A'.$$

**What does it mean for $A'$ in order that $h$ be a edge dominating signing of $G$?**

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## Dominating Signing of a Matrix

Let $X = [x_{ij}]$ be an $m$ by $n$ matrix. The **cross** $C_{p,q}$ of an entry $x_{pq}$ is the set of positions of $X$ in row $p$ and column $q$:

$$C_{pq} = \{(p,j) : 1 \leq j \leq n\} \cup \{(i,q) : 1 \leq i \leq m\}$$

$$\begin{bmatrix} & & x_{1q} & & \\ & & \vdots & & \\ x_{p1} & \cdots & x_{pq} & \cdots & x_{pn} \\ & & \vdots & & \\ & & x_{mq} & & \end{bmatrix}.$$

The **cross sum** of the entry $x_{pq}$ is the sum of the entries in its cross.

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A signing $A' = [a'_{ij}]$ of $A$ corresponds to a **edge dominating signing** of $G$ provided the sum of the entries in the cross of each **nonzero entry** of $A'$ is at least 1:

$$a'_{pq} + \sum_{j \neq q} a'_{pj} + \sum_{i \neq p} a'_{iq} \geq 1 \text{ whenever } a'_{pq} \neq 0.$$

Each cross must contain at least one more 1 than $-1$.

Such a signing of $A$ is called a **dominating signing** of $A$ (delete the word 'edge').

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# Example

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad A' = \begin{bmatrix} 1 & -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The **value** of a signing $A' = [a'_{ij}]$ of $A$ is the sum of the entries of $A'$: $\sigma(A') = \sum_{ij} a'_{ij}$.

In the example, the value is $\sigma(A') = 8$.

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# The Problem

**Given a $(0,1)$-matrix $A$, determine the minimum value of a dominating signing of $A$, the signed domination number $\gamma'_{sd}$ of $A$:**

$$\gamma'_{sd}(A) = \min\{\sigma(A') : A' \text{ is a dominating signing of } A\}.$$

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# Cycles in General

Let $n$ be an integer with $n \geq 2$, and let $P_n$ denote the full cycle permutation matrix of order $n$. Then

$$\gamma'_{sd}(I_n + P_n) = \begin{cases} \frac{2n}{3} & \text{if } 2n \equiv 0 \bmod 3, \\ \frac{2n+2}{3} & \text{if } 2n \equiv 1 \bmod 3, \\ \frac{2n+4}{3} & \text{if } 2n \equiv 2 \bmod 3. \end{cases}$$

# Xu's 2005 Conjecture for $(0,1)$-Matrices

**Conjecture:** If $A$ is an $m$ by $n$ (0,1)-matrix, then

$$\gamma'_{sd}(A) \leq m + n - 1.$$

**Example:** $A_{4,5} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

Each 1 is in a cross containing only two 1s, and so no $-1$s are possible in a dominating signing: $\gamma'_{sd}(A_{4,5}) = 8 = 4 + 5 - 1$. Thus, in this case, equality holds in the conjecture.

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# An Improved Conjecture for Regular Matrices

$\mathcal{A}(n,k)$ denotes the set of $(0,1)$-matrices of order $n$ with $k$ 1s in every row and column $(1 \le k \le n)$.

Xu's conjecture, if true, implies that $\gamma'_{sd}(A) \le 2n-1$ for $A \in \mathcal{A}(n,k)$.

**Theorem:**

$$\gamma'_{sd}(A) \le \begin{cases} n & \text{if } k \text{ is odd} \\ 2n-4 & \text{if } k \text{ is even.} \end{cases}$$

In fact, for the $n$ by $n$ matrix $J_{n,n}$ of all 1s (so $J_{n,n} \in \mathcal{A}(n,n)$), we have

$$\gamma_{sd}(J_{n,n}) = n,$$

and thus for $k$ odd, the upper bound as a function of $n$ only cannot be improved.

**Conjecture:** If $k$ is even, then $\gamma'_{sd}(A) \le \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$

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These inequalities are attainable: If $n$ is even, by $J_{n,n}$ with $k = n$ (so even), and if $n$ is odd, by the following theorem when $n$ is odd and $k = n - 1$ (so even).

**Theorem:**

$$\gamma_{sd}(J_{n,n} - I_n) = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Note that, in general, to get an exact signed domination number, it is required to obtain a dominating signing with a specified number of $-1$s and then show that no greater number of $-1$s is possible in a dominating signing.

# Remark

To indicate the sometimes delicate nature of the signed domination number of a matrix, recall that

$$\gamma'_{sd}(J_{n,n}) = n \text{ for all } n \geq 1, \text{ but}$$

**Theorem:**

- If $J^{\#}_{n,n}$ is the matrix obtained from $J_{n,n}$ by replacing a 1 with a 0, then

  $$\gamma_{sd}(J^{\#}_{n,n}) = n + 1.$$

  A signing that gives the value $n+1$ is (assuming the 0 is in the $(n,n)$ position)

  $$n \text{ even: } \begin{bmatrix} 2I_{n/2} - J_{n/2} & J_{n/2} \\ J_{n/2} & -J^{\#}_{n/2} \end{bmatrix}$$

# Remark continued

$$n \text{ odd: } -I_n^{\#} - P_n - \cdots - P_n^{(n-3)/2} + P_n^{(n-1)/2} + \cdots + P_n^{n-1}.$$

- For the $n$ by $n+1$ matrix $J_{n,n+1}$ of all 1s,

$$\gamma'_{sd}(J_{n,n+1}) = 2n.$$

A best signing is obtained by taking a best signing for $J_{n,n}$ and appending a column of all 1s.

# Remark continued

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# Lower Bound on Signed Domination Number

**Theorem:** Let $A$ be a $(0,1)$-matrix of order $n$ with $k$ 1s in each row and column. Then

$$\gamma'_{sd}(A) \geq \frac{kn}{2k-1}.$$

More generally, if $G$ is a graph of order $n$ which is regular of degree $k$, then

$$\gamma'_{sd}(G) \geq \frac{kn}{2(2k-1)}.$$

**Can equality be attained in these inequalities?** For equality to hold, all cross sums must equal 1 (the minimum possible).

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## Equality in the Lower Bound

**Theorem:** For each $k$, the minimum number $n$ of vertices in a regular graph $G$ of degree $k$ satisfying $\gamma'_{sd}(G) = \frac{kn}{2(2k-1)}$ equals

$$(2k-1)C_{k-1}$$

where $C_{k-1}$ is the $(k-1)$st Catalan number.

Moreover,

1. If $\gamma'_{sd}(G) = \frac{kn}{2(2k-1)}$, then $n$ is a multiple of $(2k-1)C_{k-1}$ and any such multiple is attainable.
2. If $G$ is bipartite, then the minimum number of vertices is

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and this is always attainable.

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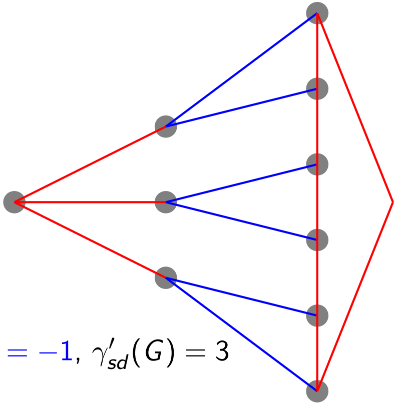
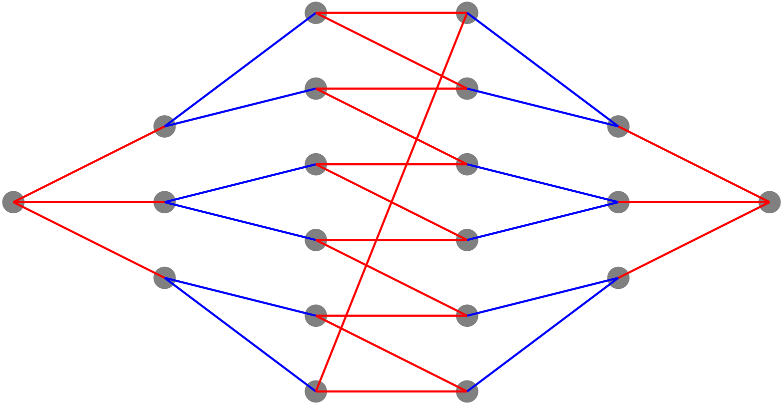
# Example of Equality in the Lower Bound ($k = 3$)



$G$: red $= +1$, blue $= -1$, $\gamma'_{sd}(G) = 3$

# Bipartite Example of Equality ($k = 3$)



$G$: red $= +1$, blue $= -1$, $\gamma'_{sd}(G) = 6$

## Bipartite Example is a Biadjacency Matrix

$k = 3$ and $n = 10$, $G \subseteq K_{10,10}$, $A = |A'|$, $\gamma'_{sd}(A) = 6$

$$A' = \left[\begin{array}{c|cccccc|ccc} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right].$$

# Semi-Regular Matrices

A **semiregular matrix** is an $m$ by $n$ (0,1)-matrix with $k$ 1s in each row and $l$ 1s in each column. Thus $km = ln$.

**Example:** $(m = 4, n = 6; k = 3, l = 2)$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Remark: Since $\gamma'_{sd}(A) = \gamma'_{sd}(A^T)$, there is no loss in generality in assuming that $m \leq n$ (and so $k \geq l$).

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# $J_{m,n}$ Part I.

Let $J_{m,n}$ denote that $m$ by $n$ matrix of all 1s (so semiregular with $k = n, l = m$).

Akbari, Bolouki, Hatami, Siami (2009) (and independently and in a somewhat different way by us) evaluated $\gamma'_{sd}(J_{m,n})$ for all $m$ and $n$. In our formulation, we have:

- If $m$ is even and $n$ is even, then

$$\gamma'_{sd}(J_{m,n}) = \begin{cases} n & \text{if } m \leq n \leq 2m-1, \\ 2m & \text{if } 2m \leq n. \end{cases}$$

- If $m$ is even and $n$ is odd, then

$$\gamma_{sd}(J_{m,n}) = \begin{cases} 2m & \text{if } m \leq n < 2m, \\ n+1 & \text{if } 2m \leq n < 3m, \\ 3m & \text{if } 3m \leq n. \end{cases}$$

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# Semi-regular Matrices: General Case I

Let $A$ be an $m$ by $n$ (0,1)-matrix with $m < n$ and $k$ 1s in each row and $l$ 1s in each column (thus $km = ln$).

**Case: $k$ and $l$ even**

$$\gamma'_{sd}(A) \leq 2m = m + m \leq m + n - 1.$$

**Case: $k$ and $l$ odd**

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# Semi-regular Matrices: General Case II

**Case:** $k$ **even,** $l$ **odd,** $n > 3m$

$$\gamma'_{sd}(A) \leq 4m \leq m + n - 1.$$

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$$\gamma'_{sd}(A) \leq n \leq m + n - 1.$$

**Case:** $k$ **even,** $l$ **odd,** $m < n \leq 2m$

$$\gamma'_{sd} A \leq 2m \leq m + n - 1.$$

**Case:** $k$ **odd,** $l$ **even,** $n > 2m$

$$\gamma'_{sd}(A) \leq 3m \leq m + n - 1.$$

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$$\gamma'_{sd}(A) \leq 3m \leq m + n - 1.$$

# Semi-regular Matrices: General Case III

**Case:** $k$ **odd,** $l$ **even,** $m < n \leq 2m$

**Unable** to show that $\gamma'_{sd} \leq m + n - 1$.

# Techniques

- Construction
- Counting
- Given a (0,1)-matrix $A$, existence of a (0,1)-matrix $B$ with $A \leq B$ and with bounds on row and column sums.
- An old theorem of Vogel concerning the maximum number of 1s one can pull out of a (0,1)-matrix $A$ without exceeding given row and column sum bounds.

# Two Problems we'd Like to Solve

1. If $k$ is odd, $l$ is even, and $m < n \leq 2m$, then

$$\gamma'_{sd} \leq m + n - 1.$$

2. If $A$ is in $\mathcal{A}(n, k)$ with $k$ even, then

$$\gamma'_{sd}(A) \leq \begin{cases} n & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd.} \end{cases}$$