Bounds on the largest families of subsets with forbidden subposets

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Congratulations to IPM, our younger brother, on its 20th birthday!

On behalf of the Renyi Institute. (We are almost 60.)

Congratulations to IPM, our nephew, on its 20 birthday!

On behalf of the Renyi Institute. (We are almost 60.)

Congratulations to Gholamreza B. Khosrovshahi on his 70th birthday

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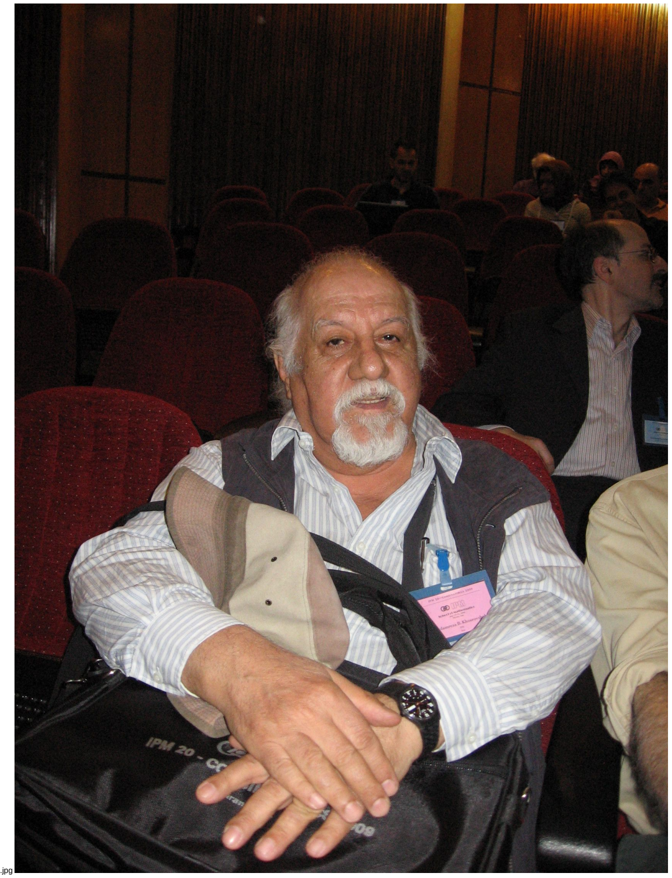
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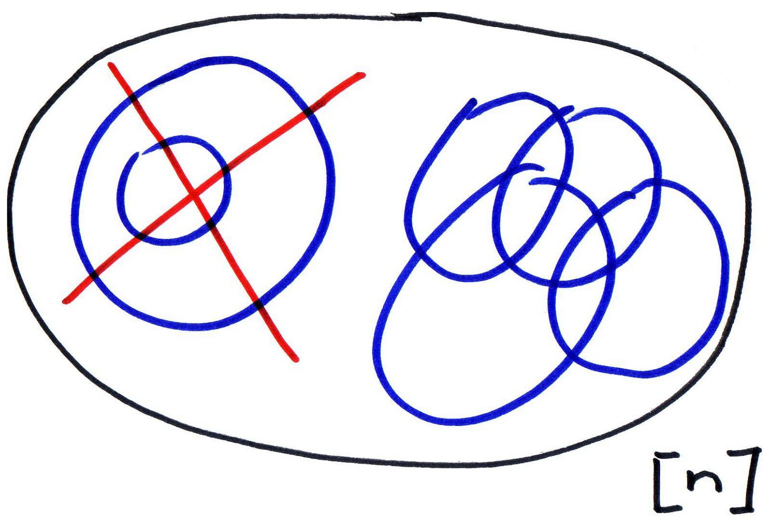
on behalf of all Hungarian combinatorialists.

The ancient combinatorial problem

Let $[n] = \{1, 2, \ldots, n\}$ be a finite set.

Question.

Find the maximum number of subsets A of $[n]$ such that $A \not\subset B$ holds for them.

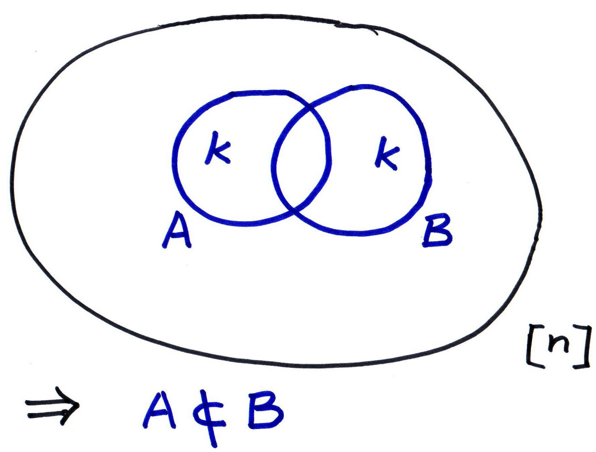


A construction

All k-element subsets ($k > 0$ is fixed) of $[n]$.

Notation: $\binom{[n]}{k}$, $\qquad |\binom{[n]}{k}| = \binom{n}{k}$.

Claim. This family $\mathcal{A}$ of subsets of $[n]$ has no inclusion.



This is largest for $k = \lfloor \frac{n}{2} \rfloor$.

Theorem (Sperner, 1928)

If $\mathcal{A}$ is a family of distinct subsets of X $(|X| = n)$ without inclusion ($A, B \in \mathcal{A}$ implies $A \not\subset B$) then

$$|\mathcal{A}| \leq \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor}.$$

Partially ordered set (Poset)

$$P = (X, <)$$

where

(i) at most one of $<, =, >$ holds,

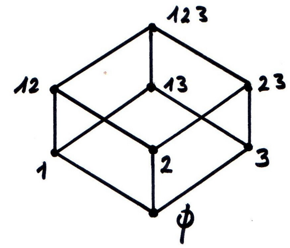
(ii) $<$ is transitive.

($a, b \in X$ are comparable in P iff $a < b, a = b$ or $a > b$.)

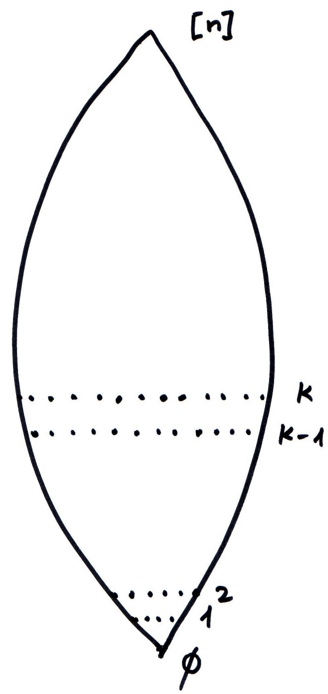
In our case $X = 2^{[n]}$ and $A < B$ in this poset iff $A \subset B$.

Notation: $B_n = (2^{[n]}, \subset)$.

B_3



B_n



Generalizations of Sperner theorem

Certain types only: when the restrictions are expressed by inclusions

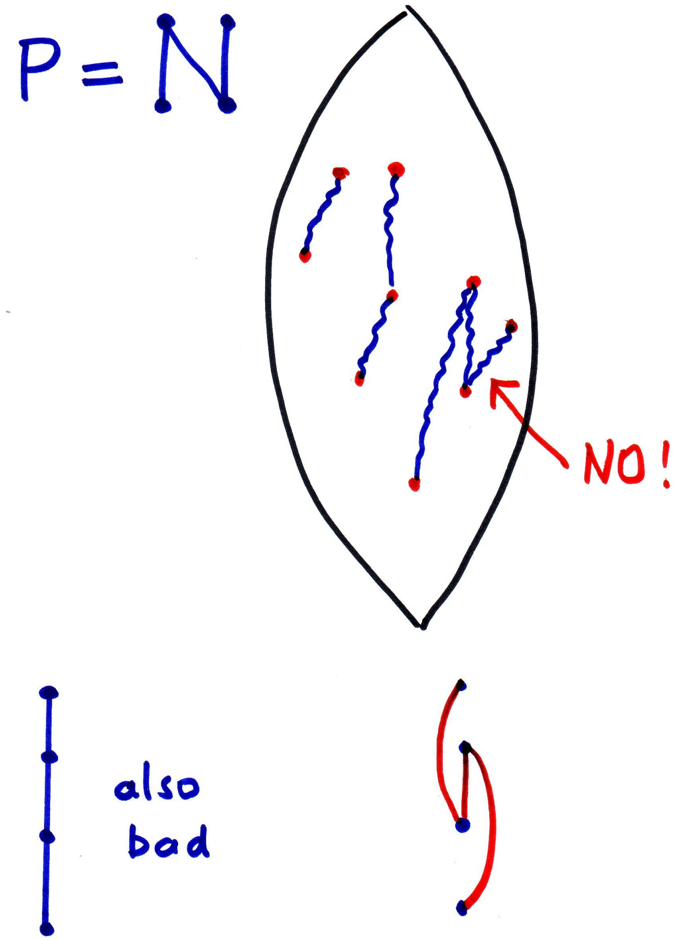
Notation. $\mathrm{La}(n, P) =$

the maximum number of elements of B_n such that the poset induced by these elements does not contain P as a subposet

short versions

=the maximum number of elements of B_n without having a copy of P

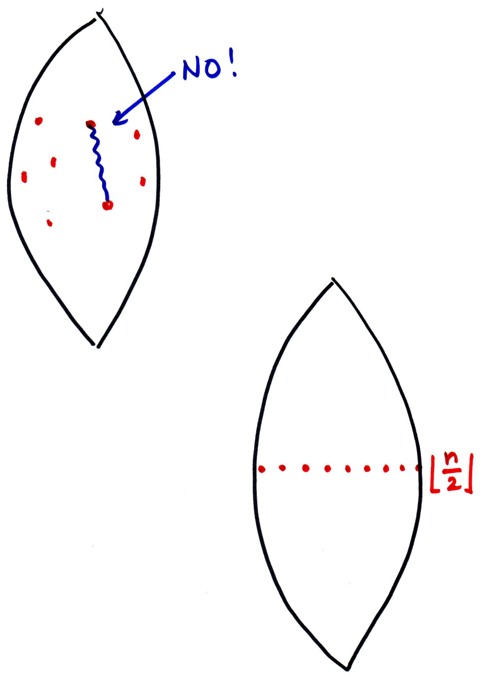
=the maximum number of subsets of an n-element set without a configuration P.


P =
NO!
also bad

Example 1: $P = I$

Theorem (Sperner)

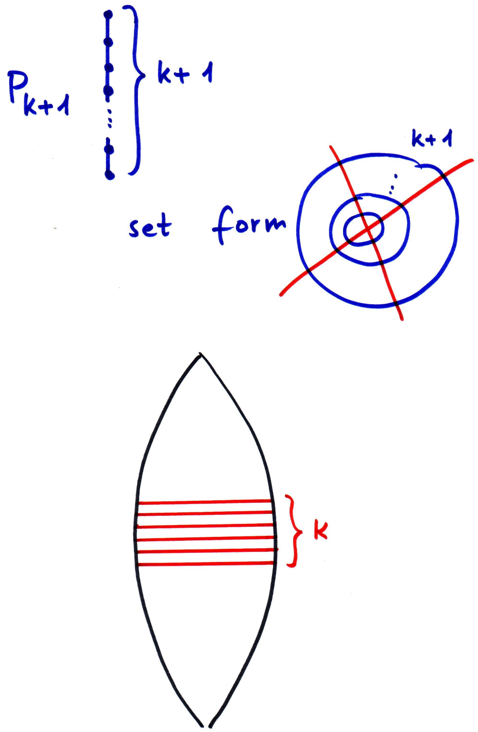
$\mathrm{La}(n, I) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.



Example 2: $P = P_{k+1}$

Theorem (Erdős, 1938)

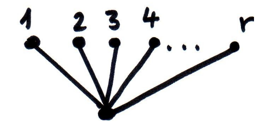
$\mathrm{La}(n, P_{k+1}) = \sum k$ largest binomial coefficients of order n.



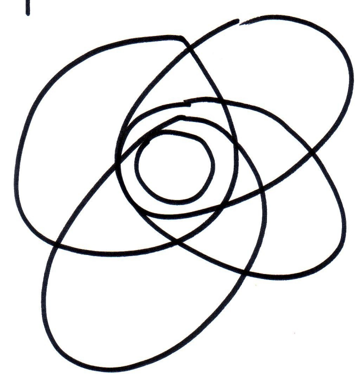
Example 3:

$$V_r = \{a, b_1, \ldots, b_r\} \quad \text{where} \quad a < b_1, \ldots a < b_r$$

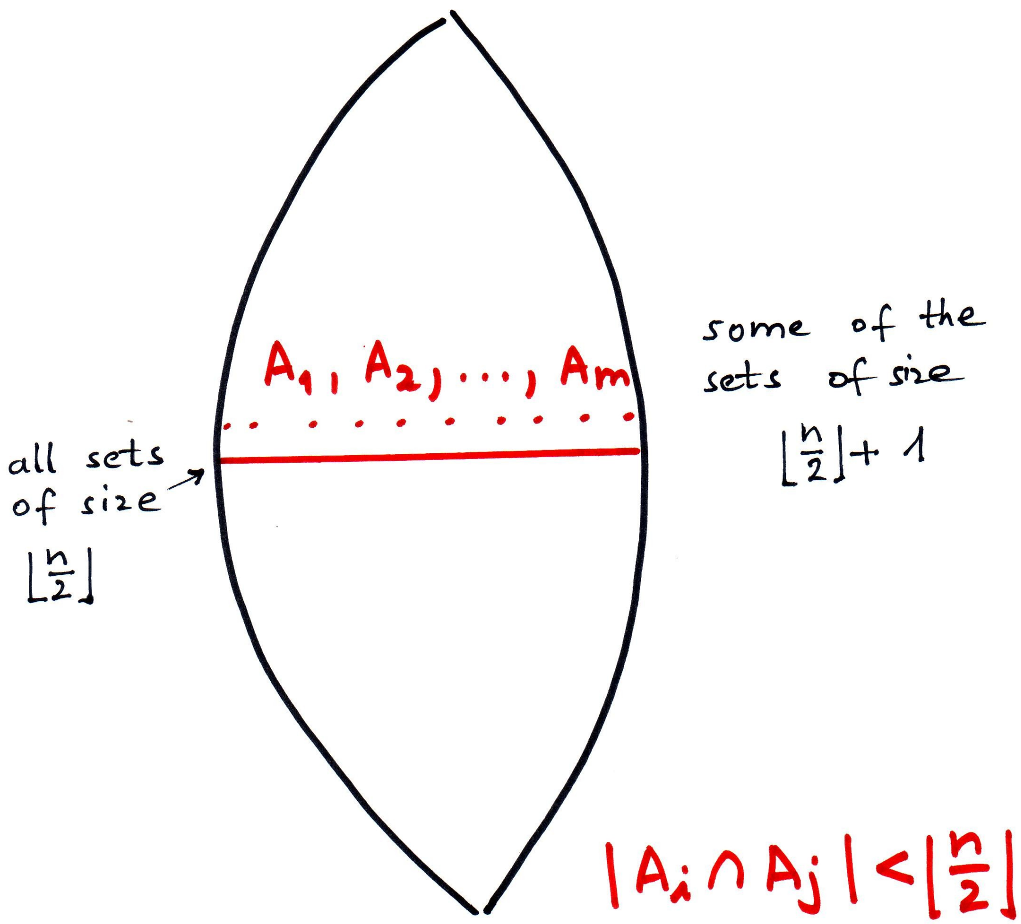
V_r



set form



Construction for V_2.



$A_1, \ldots, A_m$ such that $|A_i \cap A_j| < \left\lfloor \frac{n}{2} \right\rfloor$ $(i \neq j)$.

This is **an old open problem of coding theory**

$$\binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} \left(\frac{1}{n} + \Omega \left(\frac{1}{n^2} \right) \right) \leq \max\ m \leq \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} \left(\frac{2}{n} + O \left(\frac{1}{n^2} \right) \right).$$

Right constant?

Theorem (K-Tarján, 1983)

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{1}{n}+\Omega\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(n, V_2) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2}{n}+o\left(\frac{1}{n}\right)\right).$$

Hard to find the right constant.

Theorem (De Bonis-K, 2007)

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{r}{n}+\Omega\left(\frac{1}{n^2}\right)\right) \leq \mathrm{La}(n, V_{r+1}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2r}{n}+O\left(\frac{1}{n^2}\right)\right).$$

Theorem (Griggs-K, 2008)

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{1}{n}+\Omega\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(n,N) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right).$$

Remark. The estimates, up to the first two terms are the same as for V_2.

A method in a form of a general theorem

$\mathcal{P}$ the set of forbidden subposets

$\mathcal{Q} = \mathcal{Q}(\mathcal{P})$ set of possible components

Example $\mathcal{P} = \{I\},\ \mathcal{Q}(I\ ) = \{P_1\}$

If $Q \in \mathcal{Q}$ let Q_n^* be a realization of Q in the Boolean lattice B_n, notation: $Q \to Q_n^*$

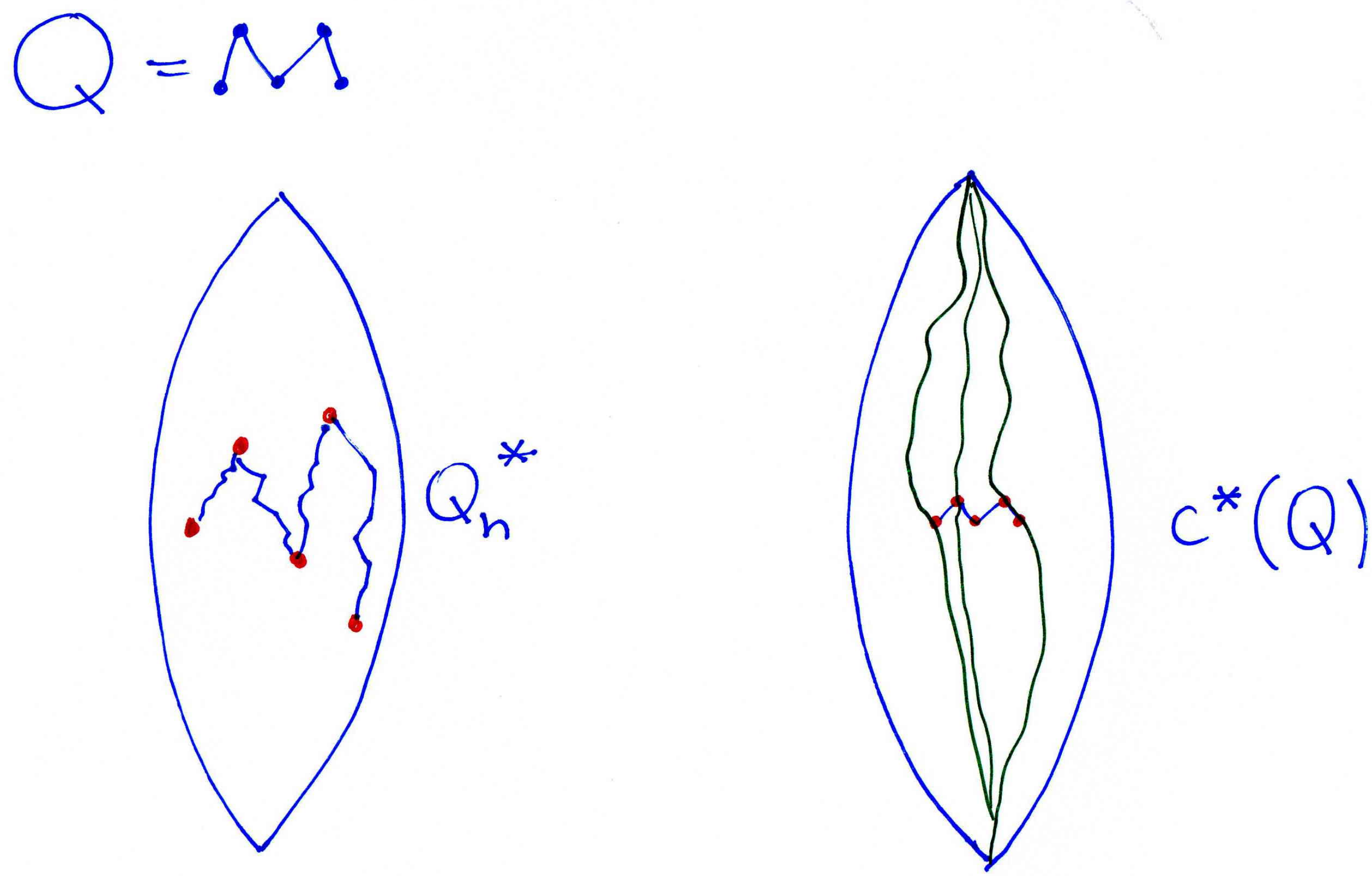
Example cont. Q_n^* is a subset (say of a elements)

$c(Q_n^*)$ number of chains going through Q_n^*

Example cont. $c(Q_n^*) = a!(n-a)!$

$\min_{Q \to Q_n^*} c(Q_n^*) = c^*(Q)$

Example cont. $c^*(Q_n^*) = \min_a a!(n-a)! = \lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil !$



Theorem

$$\mathrm{La}(n,\mathcal{P}) \leq \frac{n!}{\inf_{Q\in\mathcal{Q}} \frac{c^*(Q)}{|Q|}}$$

Example $\mathcal{P} = \{V_2\},\ \mathcal{Q}\{(V_2\}) = \{P_1 = \Lambda_0, P_2 = \Lambda_1, \Lambda_2, \ldots, \Lambda_r, \ldots\}$

Unbounded number of possible types of components!

Claim:

$$u^*!u^*(n-u^*-1)! \leq \frac{c^*(\Lambda_r)}{r+1}$$

where $u^* = u^*(n) = \frac{n}{2} - 1$ if n is even, $u^* = \frac{n-1}{2}$ if n is odd and $r - 1 \leq n$, while $u^* = \frac{n-3}{2}$ if n is odd and $n < r - 1$.

$$\mathrm{La}(n, V_2) \leq \frac{n!}{u^*!u^*(n-u^*-1)!} = \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Next example $\mathcal{P} = \{N\},\ \mathcal{Q}\{(N\}) = \{P_3, V_r(0 \le r), \Lambda_r(0 \le r)\}$

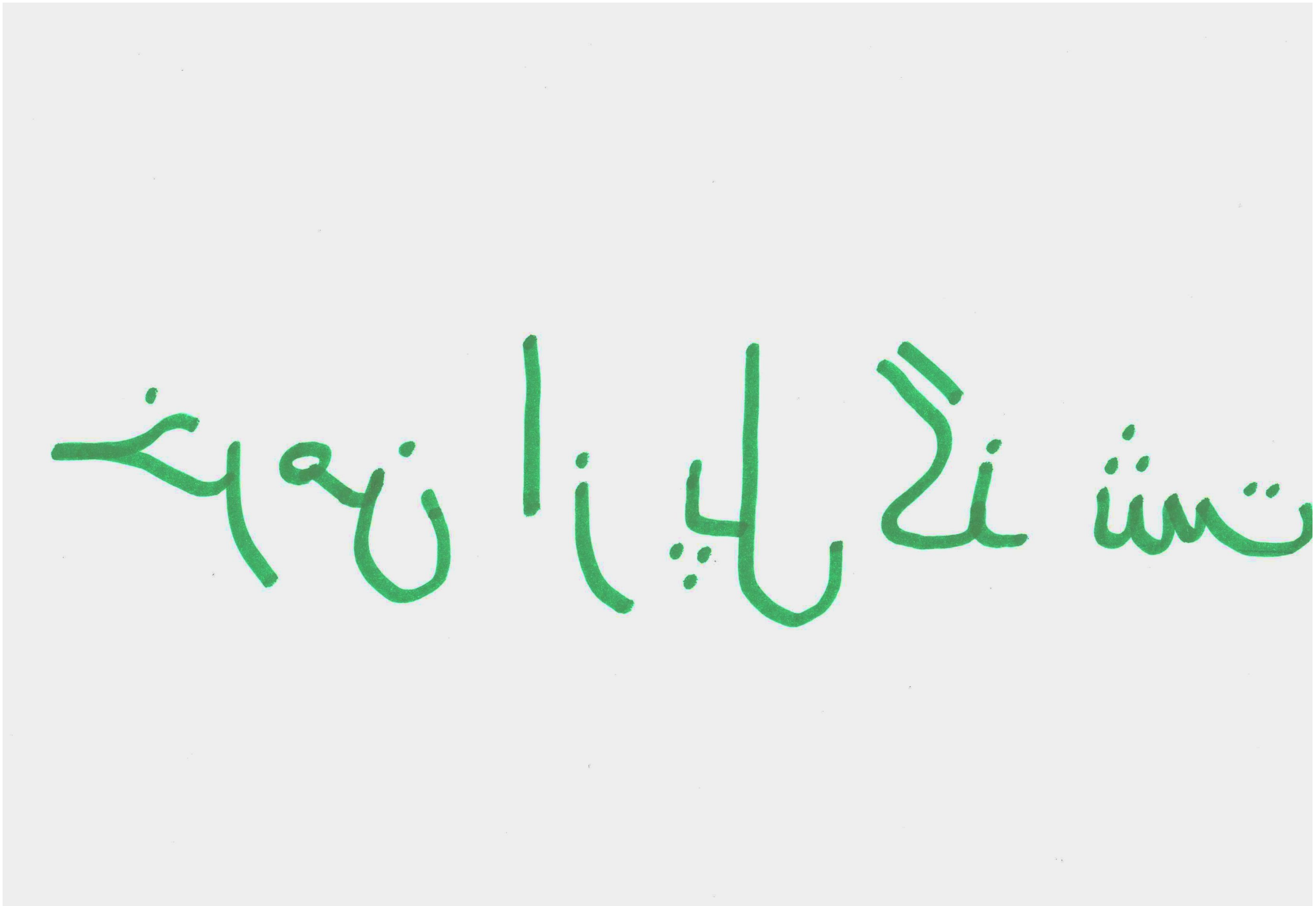
The only novelty needed here:

$$u^*!u^*(n - u^* - 1)! \le \frac{c^*(P_3)}{3}$$

Then, again,

$$\mathrm{La}(n, N) \le \frac{n!}{u^*!u^*(n - u^* - 1)!} = \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right).$$

خرمن از پل گذشت



But!

It is interesting to mention that the “La” function will jump if the excluded poset contains one more relation. The *butterfly* $\bowtie$ contains 4 elements: a, b, c, d with $a < c, a < d, b < c, b < d$.

Theorem (De Bonis, K, Swanepoel, 2005) Let $n \geq 3$. Then $\mathrm{La}(n, \bowtie) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$.

Try $\mathcal{P} = V_3$

If $Q \in \mathcal{Q}$ and $x \in Q$ then at most two ”edges” can go ”upwards” from x and any number of ”edges” ”downwards”.

Q can be only on 3 levels, but can be very messy. It seems to be impossible to find the minimum of $c(Q_n^*)$ for each such Q.

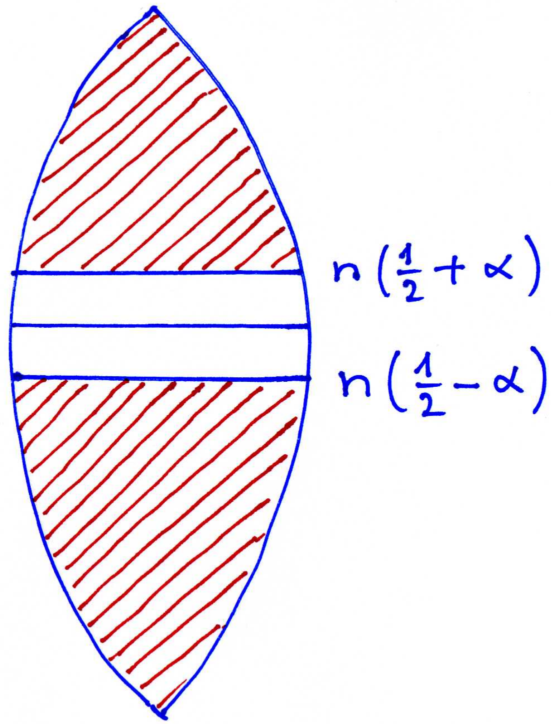
One more simple idea is needed

The main part of a large family is near the middle!

Let $0 < \alpha < \frac{1}{2}$ be fixed. The total number of sets F of size satisfying

$$|F| \notin \left[n\left(\frac{1}{2} - \alpha\right), n\left(\frac{1}{2} + \alpha\right) \right] \qquad (\bullet)$$

is very small.



The shaded part is small in comparison with 2^n.

Theorem Let $0 < \alpha < \frac{1}{2}$ be a real number. Then

$$\mathrm{La}(n, \mathcal{P}) \leq \frac{n!}{\inf_{Q \in \mathcal{Q}(\mathcal{P})} \frac{c_n^{*\alpha}(Q)}{|Q|}} + \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^2}\right).$$

Here α means that we forget about elements in the shaded area.

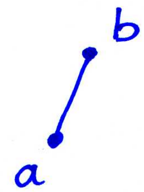
Use the sieve!

The number $c(Q_n^*)$ can be estimated from below with the first two terms of the sieve:

$$c(Q_n^*) \geq \sum (\text{the number of chains going through a point of } Q_n^*) -$$
$$\sum (\text{the number of chains going through two given points of } Q_n^*).$$

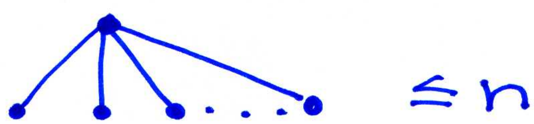
In many cases one can easily find the (approximate) minimum of this sum under the condition that the sets are **around the middle**. **Otherwise it is rough.**

b **covers** a if $a < b$ and there is no c such that $a < c < b$



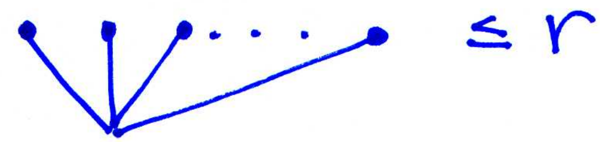
b covers a

Theorem Let $1 \leq r$ be a fixed integer, independent on n. Suppose that every element $Q \in \mathcal{Q}(\mathcal{P})$ has the following property: if $a \in Q$ then a **covers** at most r elements of Q.



Then

$$\mathrm{La}(n, \mathcal{P}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

no $V_{r+1} \Rightarrow$ 

The result for La(n, V_{r+1}) is a **consequence**.

Excluding induced posets, only

Now we exclude the posets P only in a **strict** form, that is, there is **no induced copy** in the poset induced in B_n by the family. La$^\sharp(n, P)$ denotes the maximum number of subsets of $[n]$) such that P is not an induced subposet of the poset spanned by $\mathcal{F}$ in B_n. For instance, calculating La(n, V_2) the path of length 3, P_3 is **also excluded**, while in the case of La$^\sharp(n, V_2)$ this is **allowed**, three sets A, B, C are excluded from the family only when $A \subset B, A \subset C$ but B and C are incomparable.

Theorem Let $1 \leq r$ be a fixed integer. Suppose that every element $Q \in \mathcal{Q}^\sharp(\mathcal{P})$ has the following property: if $a \in Q$ then a covers at most r elements of Q. Then

$$\text{La}^\sharp(n, \mathcal{P}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

A consequence

Theorem

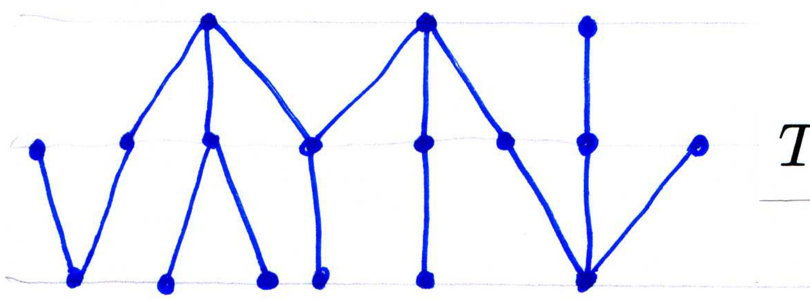
$$\mathrm{La}^\sharp(n, V_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

The **special case** $r = 1$ was solved in a paper of Carroll-K (2008).

Results for trees

A poset is a tree if the graph defined by covering pairs is a tree.

T with 2 levels



T with 3 levels, $h(T) = 2$

Results for trees

Theorem (Griggs-Linyuan Lincoln Lu, 2008+) Let T be a tree and suppose that it has two levels, then

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\Omega\left(\frac{1}{n}\right)\right) \leq \mathrm{La}(n,T) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+O\left(\frac{1}{n}\right)\right).$$

Let $h(P)$ denote the hight (maximal length of a chain) in a poset.

Theorem (Bukh, 2008+) Let T be a tree. Then

$$h(T)\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\Omega\left(\frac{1}{n}\right)\right) \leq \mathrm{La}(n,T) \leq h(T)\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+O\left(\frac{1}{n}\right)\right).$$

Some more new results

Let $G = (V, E)$ be a graph. $P(G)$ is the poset on two levels, V is the level below, $v < e (v \in V, e \in E)$ iff $v \in e$.

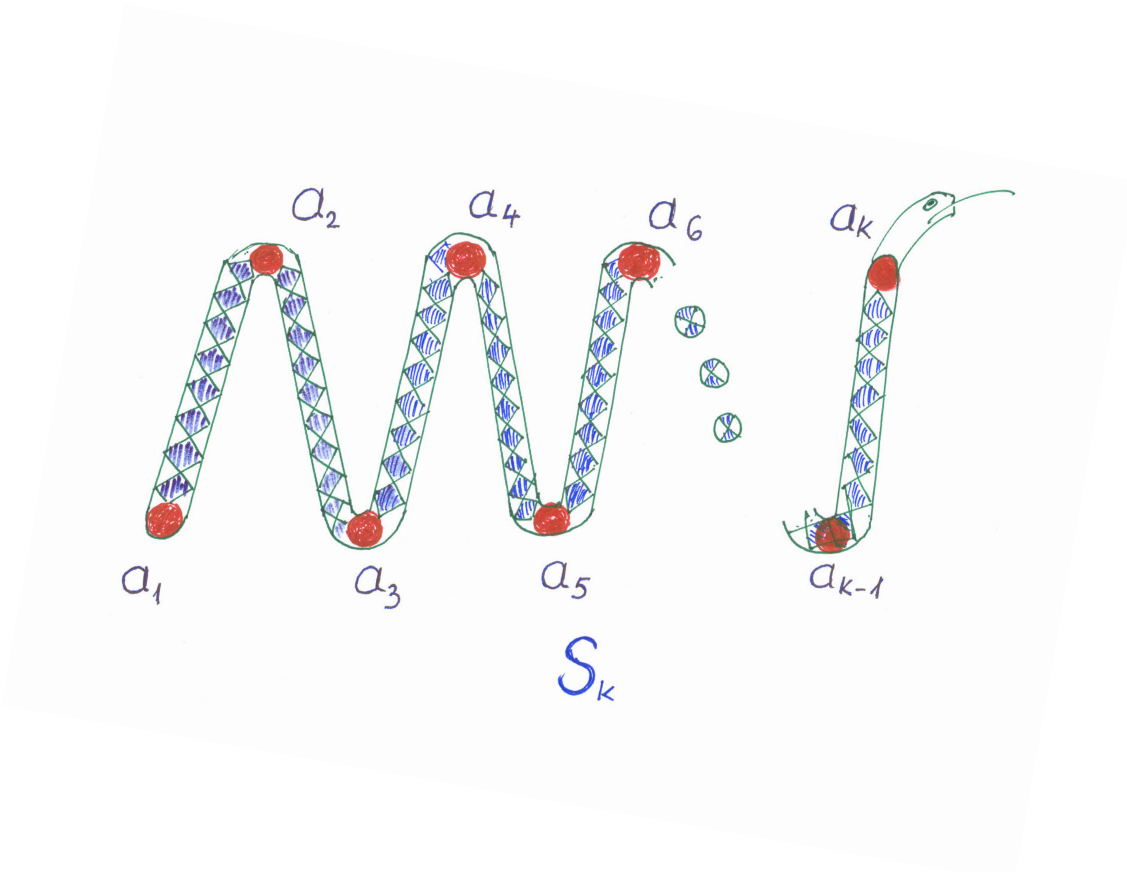
Theorem (Griggs, Linyuan Lu, 2008+) If G is bipartite then

$$\mathrm{La}(n, P(G)) \leq (1 + o(1)) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Theorem (K, 2008+)

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq La(n, M) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right)$$

The *k*-snake S_k is $a_1 < a_2 > a_3 < \ldots a_k$.



We have the same upper bound (up to the second term)

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right)$$

for $V_2 = S_3, N = S_4, M = S_5$. Is it true for general k?

We have the same upper bound (up to the second term)

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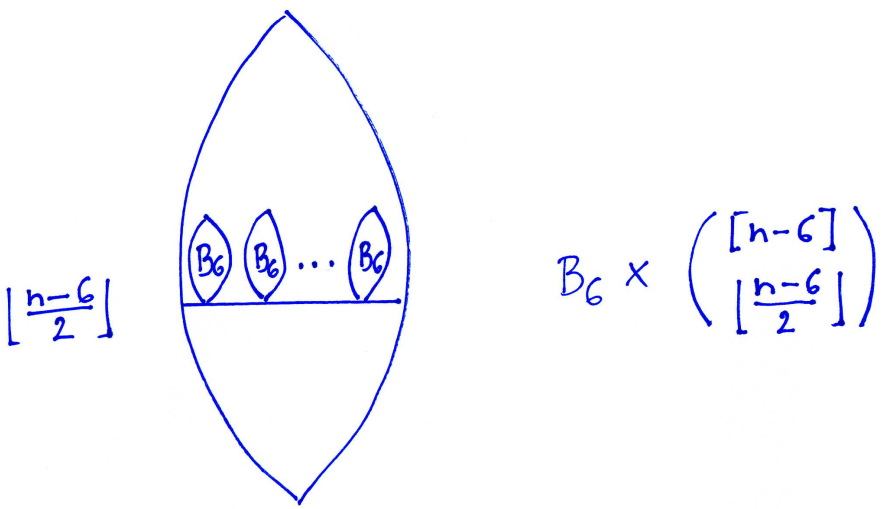
for $V_2 = S_3, N = S_4, M = S_5$. Is it true for general k?

NO!

Construction not containing S_{65}.



is a union of many disjoint copies of B_6.

NO!

Its size is (n is even)

$$64\binom{n-6}{\frac{n-6}{2}} = \binom{n}{\frac{n}{2}}\left(1 + 9\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right).$$

It is probably not true for S_6.


تشکر میکنم