Geometric distance-regular graphs

J. Koolen

Department of Mathematics
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Outline

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Distance-regular graphs

Definition

- $\Gamma_i(x) := \{y \mid d(x,y) = i\}$

Definition

- A connected graph Γ is called **distance-regular** (DRG) if there are numbers a_i, b_i, c_i $(0 \leq i \leq D = D(\Gamma))$ s.t. if $d(x,y) = j$ then
  - $\#\Gamma_1(y) \cap \Gamma_{j-1}(x) = c_j$
  - $\#\Gamma_1(y) \cap \Gamma_j(x) = a_j$
  - $\#\Gamma_1(y) \cap \Gamma_{j+1}(x) = b_j$

Outline

Γ: DRG with diameter D.

- Γ is b_0-regular. ($k := b_0$ is called its valency).
- $1 = c_1 \leq c_2 \leq \ldots \leq c_D$.
- $b_0 \geq b_1 \geq \ldots \geq b_{D-1}$.
- $b_i + a_i \geq a_1 + 1$.
- $c_i + a_i \geq a_1 + 1$.

- Define $k_i := \#\Gamma_i(x)$ $(x \in V)$.
- k_i does not depend on x.
- $(k_i)_i$ is an **unimodal** sequence.
- $v := 1 + k_1 + \ldots + k_D$: number of vertices.

Outline

Hamming graphs

Definition

- $q \geq 2$, $n \geq 1$ integers.
- $Q = \{1, \ldots, q\}$
- Hamming graph $H(n, q)$ has vertex set Q^n
- $\mathbf{x} \sim \mathbf{y}$ if they differ in exactly one position.
- Diameter equals n.
- $H(n, 2) = n$-cube.
- DRG with $c_i = i$.
- This is an example of regular near polygon (A DRG without induced $K_{2,1,1}$'s such that $a_i = c_i a_1$. for all i.)

Johnson graphs

Definition

- $1 \leq t \leq n$ integers.
- $N = \{1, \ldots, n\}$
- Johnson graph $J(n,t)$ has vertex set $\binom{N}{t}$
- $A \sim B$ if $\#A \cap B = t - 1$.
- $J(n,t) \approx J(n, n-t)$, diameter $\min(t, n-t)$.
- DRG with $c_i = i^2$.

Outline

Conjecture (Bannai-Ito(1984))

For given $k \geq 3$ there are only finitely many DRG with valency k.

- $k = 1$: K_2.
- $k = 2$: the n-gons.
- $k = 3$: (Biggs, Boshier, Shawe-Taylor (1986)) 13 DRG, diameter at most 8.
- $k = 4$: (Brouwer-K.(1999)); 12 intersection arrays: diameter at most 7.
- Note that to show the conjecture we only need to show that the diameter is bounded by a function in k.

Outline

Ivanov Bound

This is joint work with Sejeong Bang (Busan) and Vincent Moulton (Norwich).

- Γ: DRG with diameter $D \geq 2$.
- $h := \#\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$: head of Γ.
- Meaning: h is about half the girth if $a_1 = 0$.

Ivanov Diameter Bound (Ivanov (1983))

Γ:DRG with diameter $D \geq 2$ and valency k. Then $D \leq h4^k$.

So to solve the Bannai-Ito Conjecture one only needs to bound the parameter h in terms of k.
This is done using the eigenvalues of the DRG. This is not possible for $k = 2$ as there are infinitely many polygons.

Outline of proof

- Find a good interval I with the following two properties:
  - We can approximate well the multiplicities of eigenvalues inside I. Note that to approximate multiplicities we can use the theory of 3-term recurrences.
  - There are at least Ch eigenvalues in I
- This will show that any two eigenvalues in I which are algebraic conjugates are very close to each other.
- Using properties of algebraic integers (interlacing alone does not give enough information) we can find a lower bound on the (total) number of algebraic conjugates (which are also eigenvalues) of eigenvalues in I.
- Finally the Ivanov diameter bound gives an upper bound on the total eigenvalues. This will give a contradiction with the lower bound in last item.

Outline

- Γ: DRG with diameter D, valency k and smallest eigenvalue θ.
- Γ is called *geometric* if Γ contains a set of cliques $\mathcal{C}$ such that
  - $\#C = -1 - \frac{k}{\theta}$ for all $C \in \mathcal{C}$; and
  - each edge xy of Γ lies in a unique clique in $\mathcal{C}$.
- Note that the cliques in $\mathcal{C}$ are *Delsarte cliques* and hence completely regular codes with covering radius $D-1$.
- This definition of geometric DRG was introduced by Godsil. It is equivalent with Bose's definition of geometric SRG when $D = 2$.

Examples

Among the examples are:

- Hamming graphs,
- Johnson graphs,
- Grassmann graphs,
- bilinear forms graphs,
- the dual polar graphs,
- regular near $2D$-gons

Some non-examples are the Doob graphs, the twisted Grassmann graphs, the Odd graphs, the halved cubes, etc.

Outline

With Sejeong Bang and Vincent Moulton we showed:

Theorem

Let $m \geq 2$ be an integer. There are only finitely many non-geometric DRG with smallest eigenvalue at least $-m$ and valency at least three.

Earlier results

- Neumaier showed it for SRG.
- Godsil show it for antipodal DRG of diameter three.
- For $m = 2$, it follows from the fact that all regular graphs with smallest eigenvalue at least -2 are line graphs, Cocktail party graphs or the number of vertices is at most 28.
- The distance-regular line graphs were classified by Mohar and Shawe-Taylor.

Outline

Let Γ be a non-geometric DRG with valency k, diameter D and smallest eigenvalue at least $-m$.

- We need to bound the diameter and the valency.
- The cases $c_2 = 1$ and $c_2 \neq 1$ behave completely different.
- for $c_2 = 1$, it is easy to bound the valency, but we need to use the Bannai-Ito conjecture to bound the diameter.
- For $c_2 \geq 2$, you first bound the diameter before you bound the valency.

Outline

Brouwer, Cohen and Neumaier asked whether every DRG with valency at least three and diameter at least three has an integral eigenvalue besides its valency. For diameter three this is known to be true.

For geometric DRG the smallest eigenvalue is integral. So the above theorem more or less answers the question by BCN.

Conjecture 1

There exists a constant C such that the degree of the minimal polynomial of the smallest eigenvalue of a distance-regular graph with valency at least three is at most C.

Among the known distance-regular graphs with valency at least three, the largest occurring degree of the minimal polynomial of any eigenvalue is three and the only DRG with three is the Biggs-Smith graph.

Conjecture 2

There exists a constant C such that the degree of the minimal polynomial of any eigenvalue of a distance-regular graph with valency at least three is at most C.

Note that if this conjecture is true it would immediately imply the Bannai-Ito Conjecture.
I think that this conjecture would be a first step towards the following conjecture of Suzuki.

Absolute bound conjecture

There exists a constant C such that the geometric girth of any DRG with valency at least three and $c_2 = 1$ is bounded by C.

Outline

Problem 1

Classify the (non-bipartite) geometric DRG with large diameter.

Neumaier showed that the geometric SRG essentially fall into two classes, namely the block graphs of Steiner systems and the so-called graphs of Latin square type. He showed:

Theorem (Neumaier)

Let $m \geq 2$ be an integer. Then there are only finitely many SRG with smallest eigenvalue $-m$, which are neither graphs of Latin square type nor the block graphs of Steiner systems.

How do we interpret this result for DRG?

Neumaier

I propose the following conjecture:

Conjecture 3

For fixed integer m at least 2, there only finitely many (non-bipartite) DRG with valency at least three and smallest eigenvalue at least $-m$, which are neither Hamming graphs, Johnson graphs, Grassman graphs nor the bilinear forms graphs.

(The Johnson and Grassman graphs play the role of the block graphs and the Hamming and bilinear forms graphs the role of the graphs of latin square type.)

- For $m = 2$, this follows from the classification of the distance-regular line graphs.
- For $m = 3$, it is true if $c_2 \geq 2$ (Bang).
- The case $m = 3$ and $c_2 = 1$ is handled by Yamazaki who considered DRG which are locally disjoint union of 3 K_{a_1+1}'s. He showed that such a graph has small diameter or it is the halved graph of a distance-biregular graph, but this does not yet show the conjecture for $m = 3$.
- In order to show it for $c_2 = 1$ one needs to bound the diameter in terms of m. This will follow from Suzuki's conjecture.

Below I will present some more evidence for the conjecture.

Let Γ be geometric with respect to $\mathcal{C}$. For x, z be vertices at distance j and $C \in \mathcal{C}$ such that $d(x, C) = i$, define

- $\psi_i = \#\{y \mid d(x, y) = i \text{ and } y \in C\}$.
- $\tau_j = \#\{D \in \mathcal{C} \mid d(x, D) = j - 1, z \in D\}$.

Then the following holds:

- ψ_i and τ_j do not depend on x, z and C, but only at the distances.
- $\psi_1 \leq \tau_2$.
- If $\psi_1 \geq 2$, then $\psi_{i+1} \geq \psi_i + 1$.
- If $\psi_1 = \tau_2 \geq 2$, then Γ is Johnson, folded Johnson or Grassman, or $k \leq g(m)$ (Ray-Chaudhuri and Sprague, Huang, Cuypers, Bang and Koolen).
- (Metsch) If Γ has classical parameters then Conjecture 3 is true.

Outline

Regular near polygons 1

For a special class of geometric DRG, namely the regular near $2n$-gons, the following is known:

Theorem

Let Γ be a regular near $2D$-gon with $D \geq 4$ and $a_1 \neq 0$.

- (Brouwer and Wilbrink, De Bruyn) If $c_2 \geq 3$, then $c_3 = c_2^2 - c_2 + 1$
- (Shult and Yanushka, Cameron) If $c_2 \geq 3$, and Γ has classical parameters, then Γ is a dual polar graph.
- If $c_2 = 2, c_3 = 3$, then Γ is a Hamming graph.

It is likely that you do not need the assumptions of classical parameters in the second item.

Regular near polygons 2

Problem 2

Classify the non-bipartite regular $2D$-gons.

Known facts:

- The geometric girth equals 4, 6, or 8, or the diameter is at most the geometric girth.
- If the diamter is at most the girth, then $k \leq (a_1 + 1)^8$.

The case with $a_1 = 1$ seems to be doable.

Outline

- Let Γ be geometric with respect to $\mathcal{C}$ and let Γ have diameter D.
- A subgraph Δ of Γ is called $\mathcal{C}$-closed if it is geodetically closed and if two vertices of $C \in \mathcal{C}$ are contained in Δ then Δ contains all vertices of C.
- The Grassman graphs, Johnson graphs, Hamming graphs, dual polar graphs and bilinear forms graphs contain for each x, y, say at distance i, a $\mathcal{C}$-closed subgraph $\Delta(x, y)$ of diameter i containing x and y.
- For regular near $2D$-gons, it is known that these $\mathcal{C}$-subgraphs exist, and they were important in classifying them.

Questions

- Find sufficient conditions for the existence of the $\mathcal{C}$-closed subgraphs $\Delta(x, y)$.
- Is Q-polynomiality a sufficient condition?
- If Γ is Q-polynomial, has $\Delta(x, y)$ dual width $D - i$ if $d(x, y) = i$?

Thank you for listening.