# Chain Partitions of Normalized Matching Posets

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*Can we partition the poset $P$ into $k$ chains with sizes $\mu_1$, ..., $\mu_k$?*

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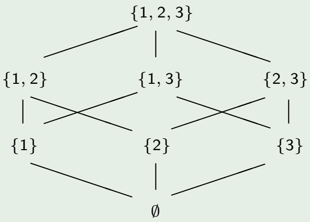

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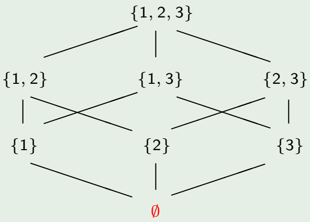

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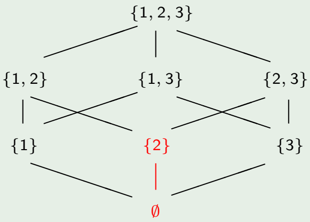

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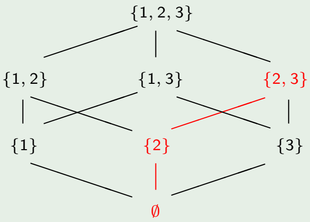

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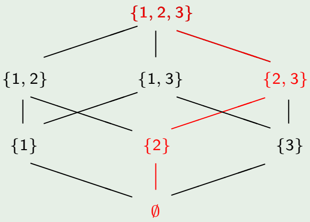

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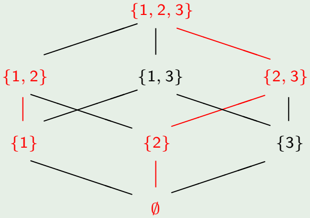

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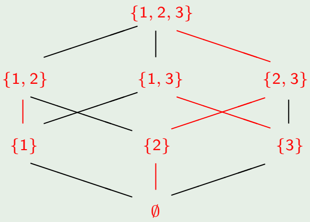

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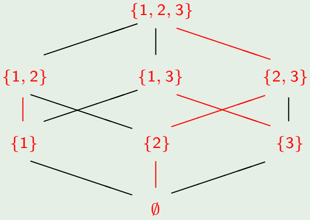

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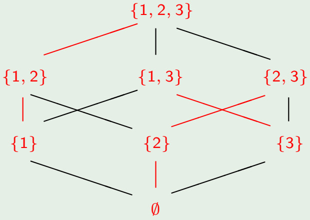


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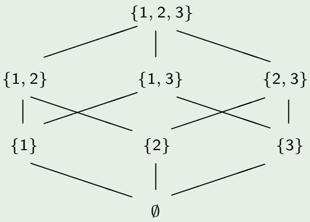


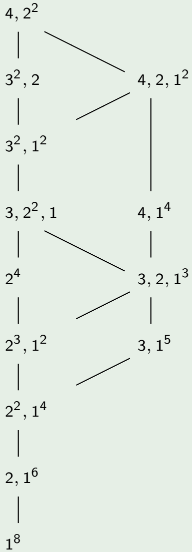

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- To focus on the so-called “Normalized Matching” or LYM posets.

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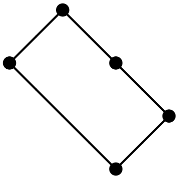
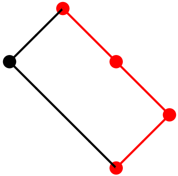
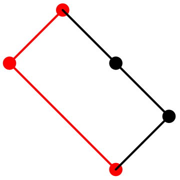
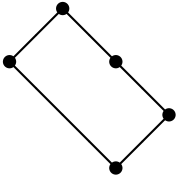
The size of the $k$-th level of $P =$ the $k$th *rank number* of $P$.

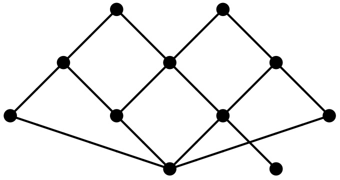


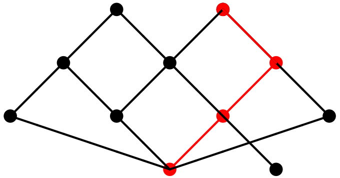


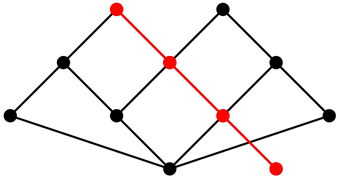




not graded







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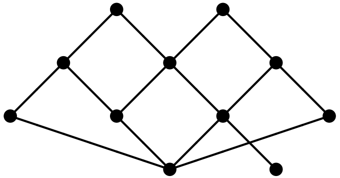

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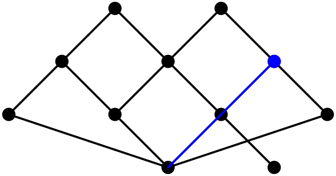

An element of rank 2

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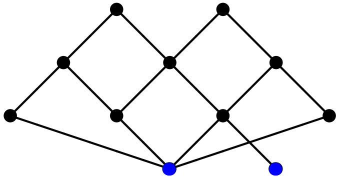

level 0: elements of rank 0

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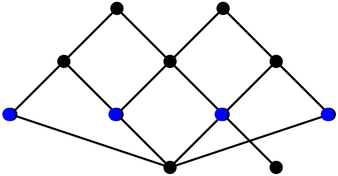

level 1: elements of rank 1

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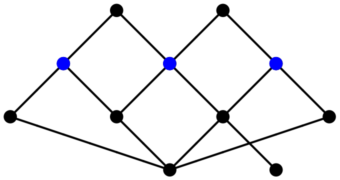

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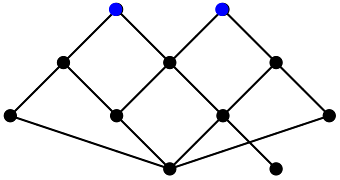

level 3: elements of rank 3

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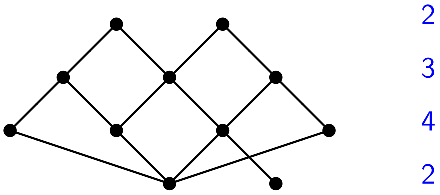


Rank Numbers: 2, 4, 3, 2

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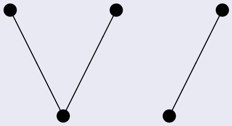

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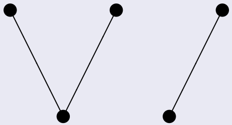

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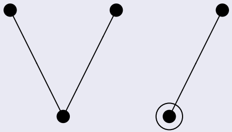

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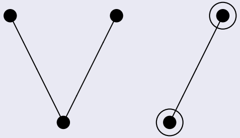

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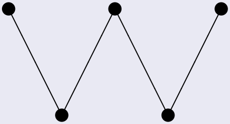

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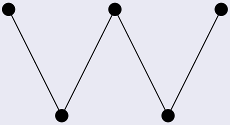
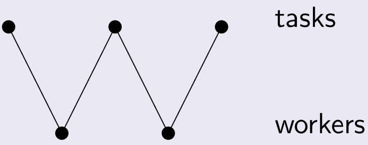


Normalized Matching

## Reformulation

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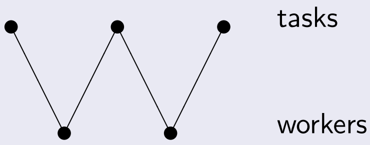


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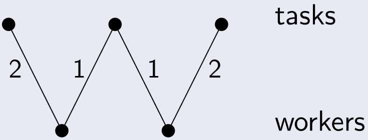


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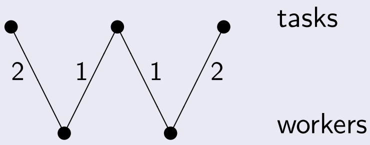


Normalized Matching

Assign each worker 3 (not nec. distinct) tasks such that each task is assigned 2 (not nec. distinct) workers.

$P$ is NM iff you can make such an assignment for each two consecutive levels.

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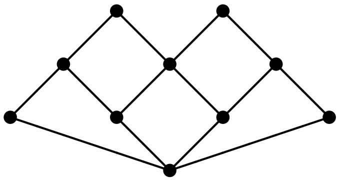
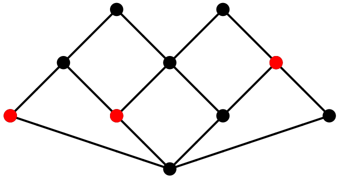
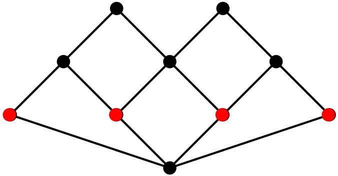
- [Dilworth's Theorem] The width of $P$ is the minimum number of chains needed to cover $P$.



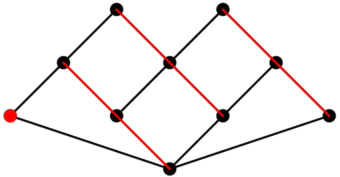


An antichain of size 3



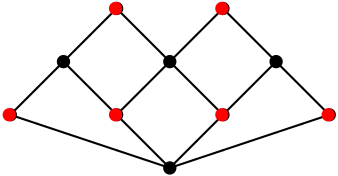
A maximum sized antichain

Width of the poset is 4



Dilworth:

Can partition into 4 chains



A 2-family

no chains of length 2:

$A < B < C$

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- *$P$ has a regular chain cover, i.e., can find a number of (not necessarily disjoint) maximal chains that cover $P$ and all the elements in the same level are on the same number of chains.*

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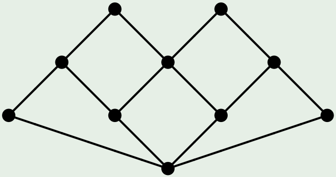

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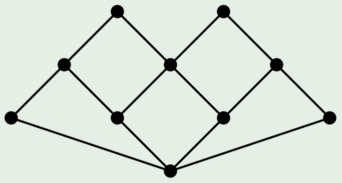

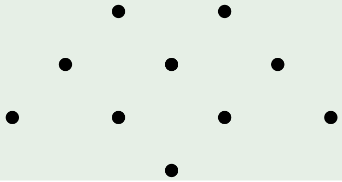

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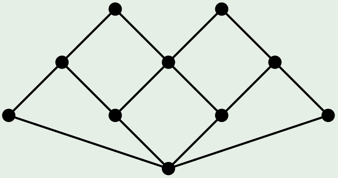

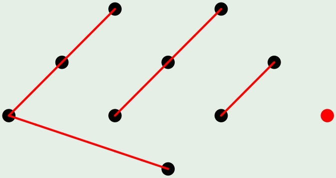

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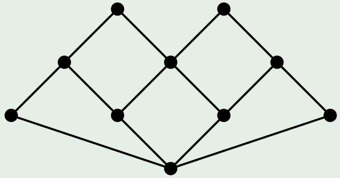

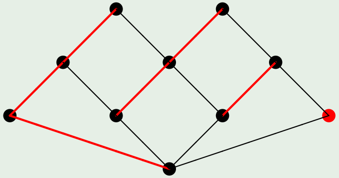

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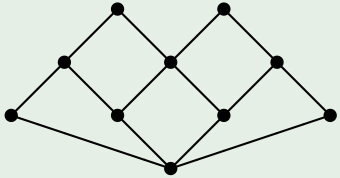

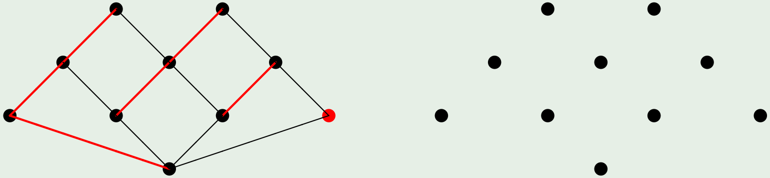

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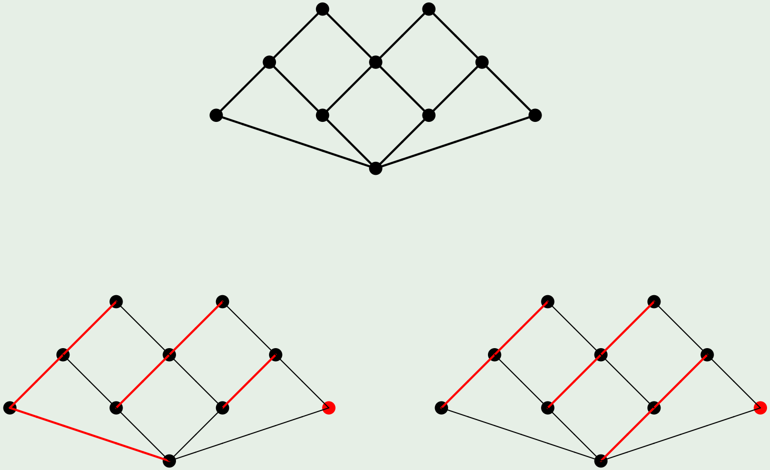

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$\sigma$ is called the *NCD partition of* $|P|$.

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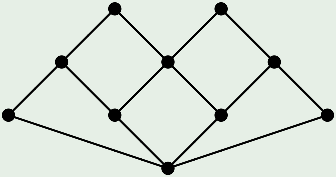

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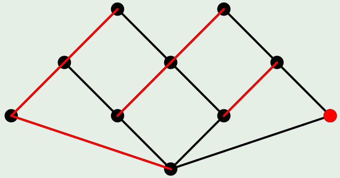

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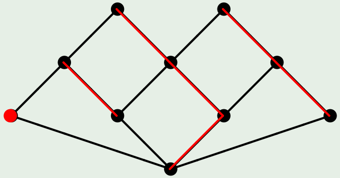

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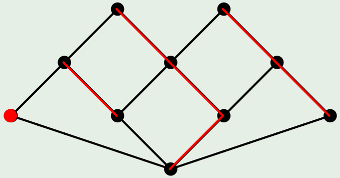

The NCD partition for any poset with rank numbers $1, 4, 3, 2$ is

$$4, 3, 2, 1.$$

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$4, 3, 1, 1$ and $5, 1, 1, 1, 1$ are incomparable.

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The only if direction is straightforward.

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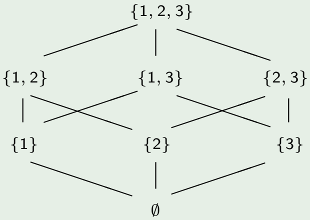
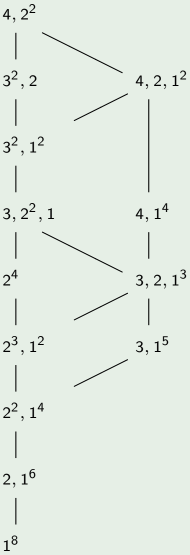

{1, 2, 3}
{1, 2}
{1, 3}
{2, 3}
{1}
{2}
{3}
∅


$4, 2^2$
$3^2, 2$
$4, 2, 1^2$
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$4, 1^4$
$2^4$
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$2^3, 1^2$
$3, 1^5$
$2^2, 1^4$
$2, 1^6$
$1^8$

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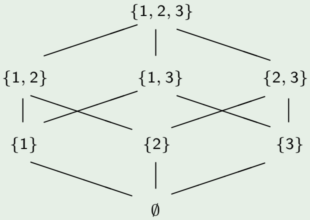
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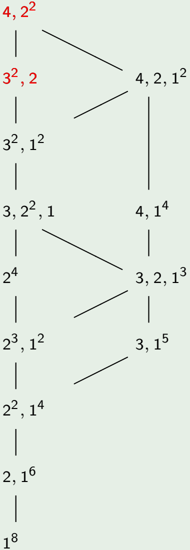
4, 2^2
3^2, 2
4, 2, 1^2
3^2, 1^2
3, 2^2, 1
4, 1^4
2^4
3, 2, 1^3
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3, 1^5
2^2, 1^4
2, 1^6
1^8

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The Füredi partition consists of only two chain sizes (two consecutive integers). It is an attempt to partition $P$ into as few chains as possible and with uniform size.

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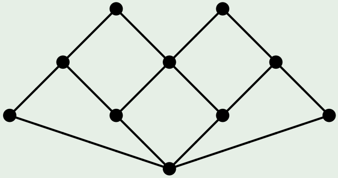

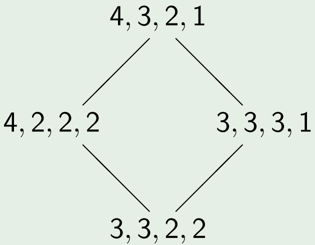

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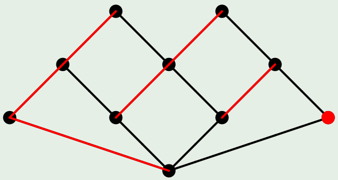

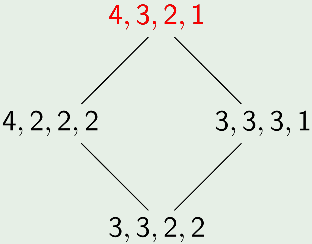

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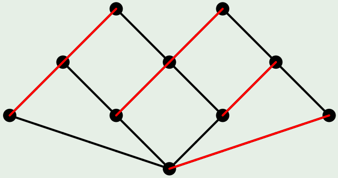

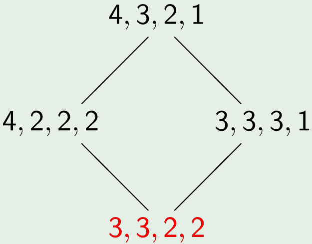

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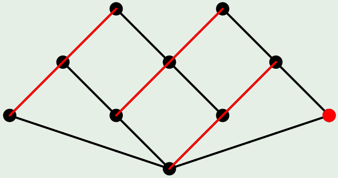

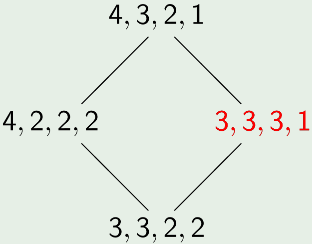

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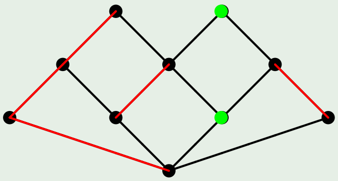

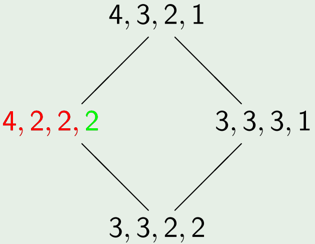

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*$\mathbf{2}^{[n]}$ can be partitioned into chains of size 4 if and only if $n \geq 9$.*

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According to the Griggs' dominating conjecture, we should be able to partition $\mathbf{2}^{[n]}$ as above for $c$ an appropriate constant multiple of $\sqrt{n}$.

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Then a Nested Chain Decomposition for $P$ is called a *Symmetric Chain Decomposition*.

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## Corollary

*Every rank 2 normalized matching poset is nested.*

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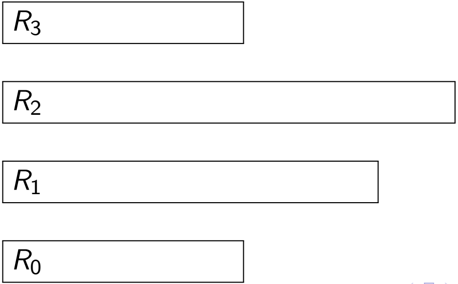

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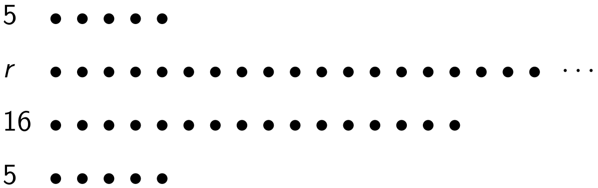

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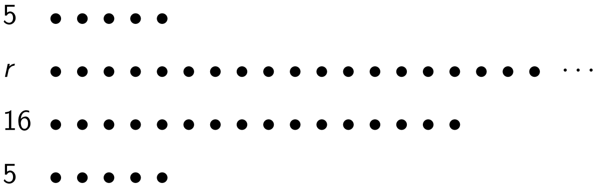


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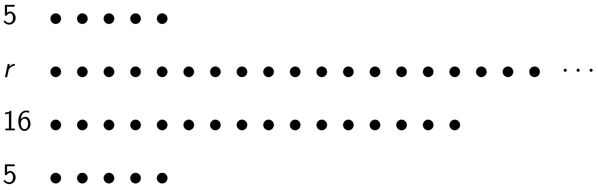


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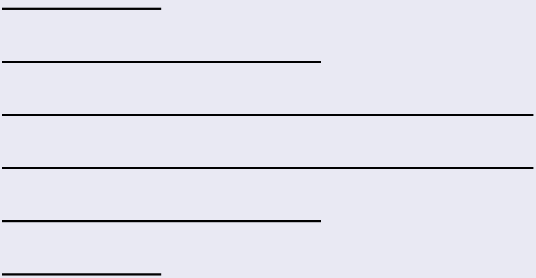

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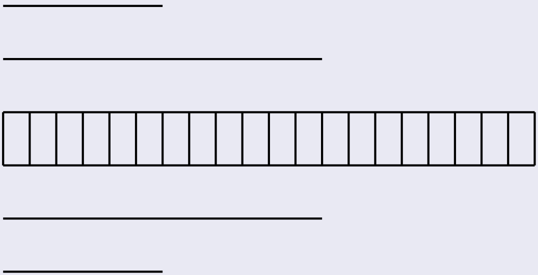

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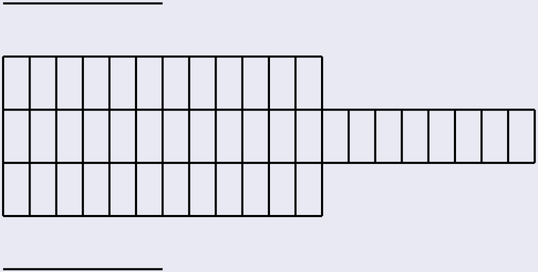

Before our work the only case (from among $\mathcal{NM}(5, 16, r, 5)$) done was the case $r = 16$.

**Theorem (Anderson 1967, Griggs 1977)**

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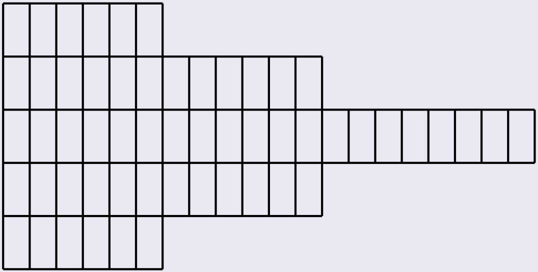

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We have shown that for $r = 20$ and 21, the matching between the middle two levels, *given by this partition*, can be extended outward to get a nesting!

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