Regular maps on a given surface – a survey

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Joint work with M. Conder, R. Nedela and T. Tucker

Introduction

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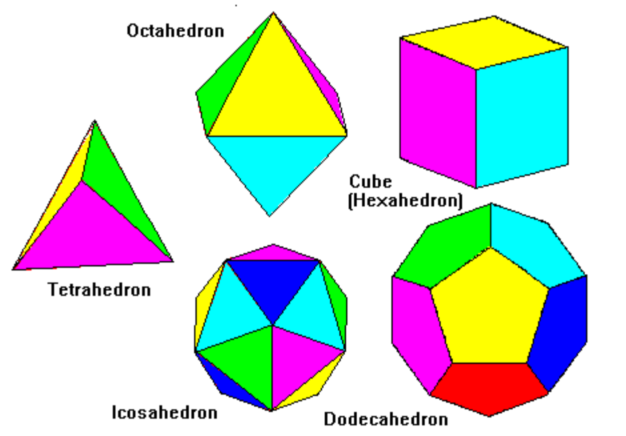

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(transitive and free action = regular action)

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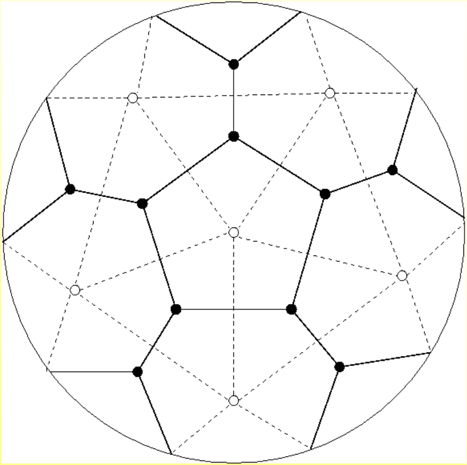

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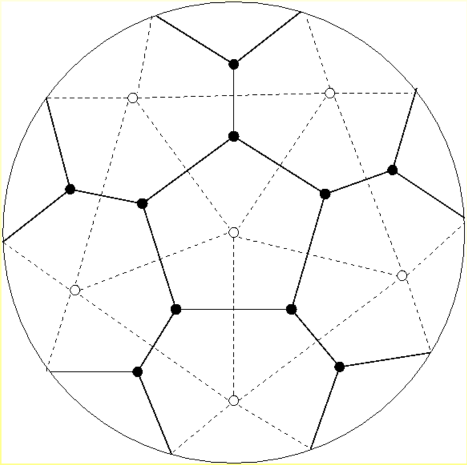

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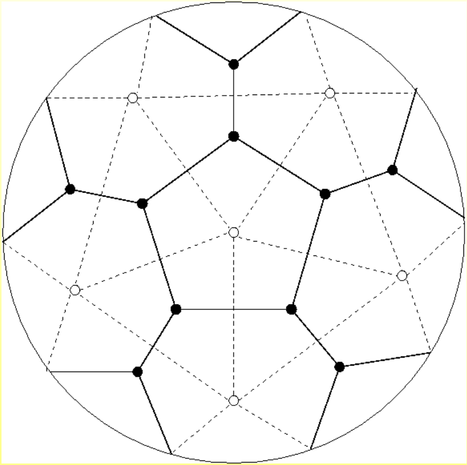

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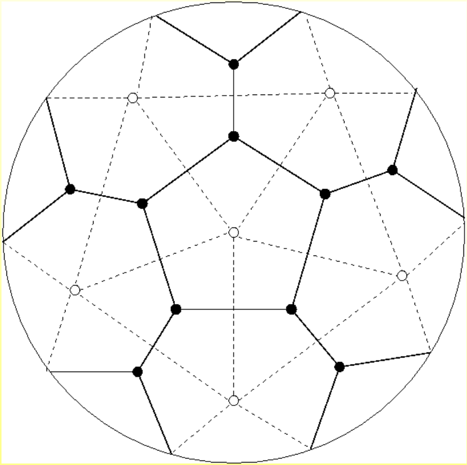

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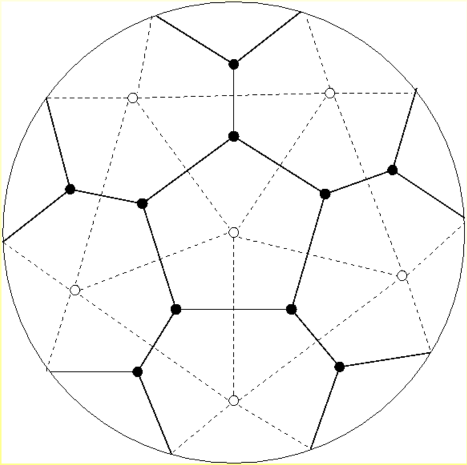

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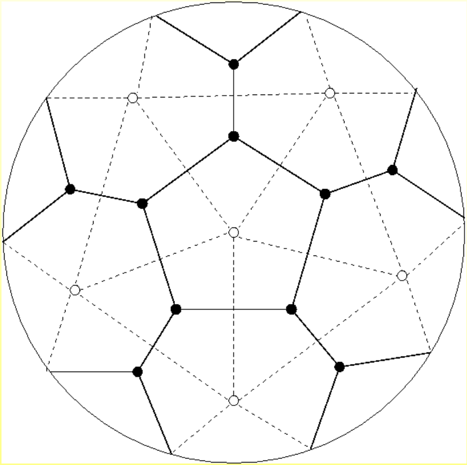

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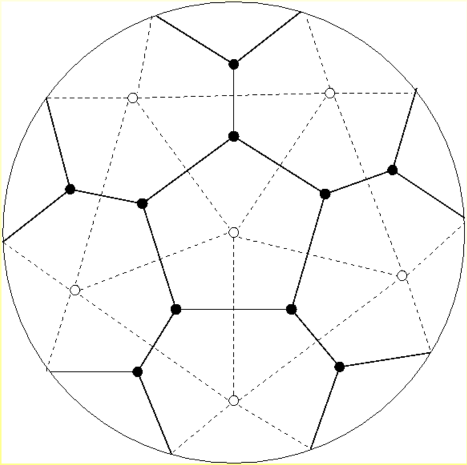

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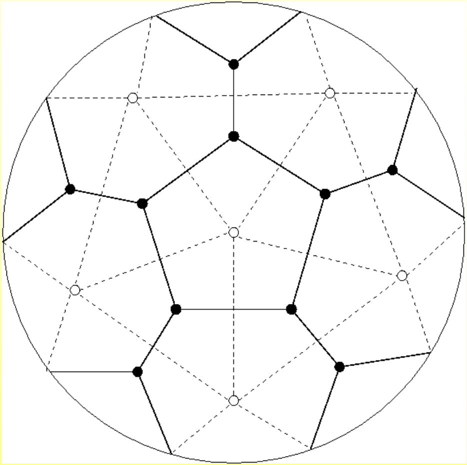

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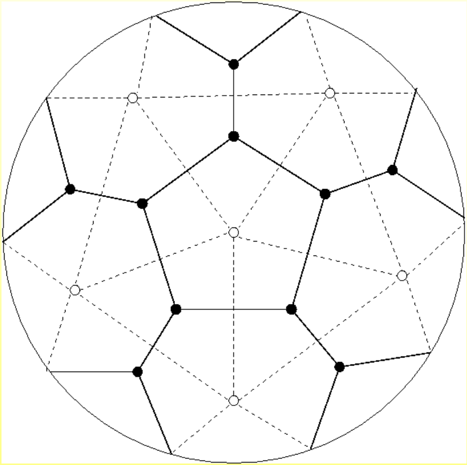

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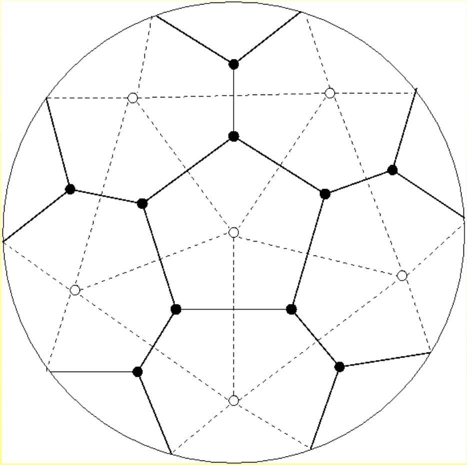

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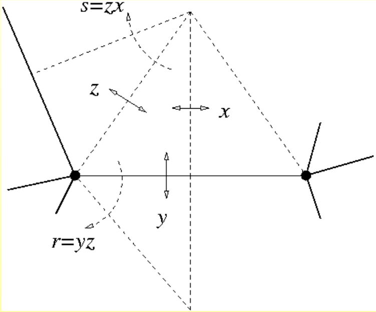

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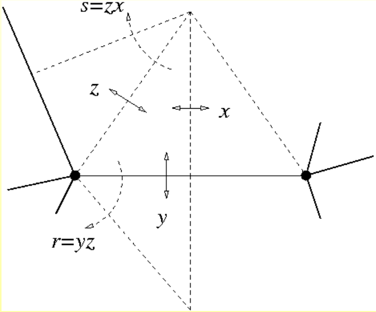

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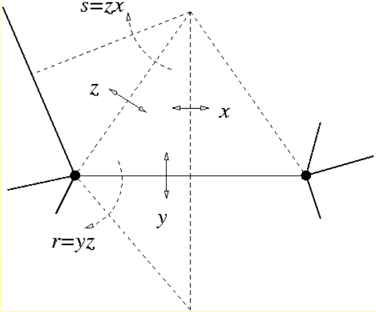

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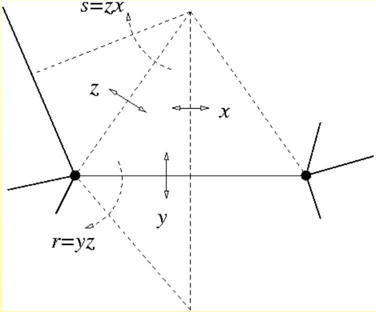

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Orientably regular maps: $Aut^o(M) = \langle r, s |\ r^k = s^m = (rs)^2 = \ldots = 1\rangle$

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In the orientably regular case we have similar one-to-one correspondences, this time with respect to *oriented triangle groups* $T^{o}(k, m, 2) = \langle r, s |\ r^k = s^m = (rs)^2 = 1\rangle$.

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The absolute Galois group can be studied
via its action on (orientably regular) maps. [Grothendieck 1981]

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Belolipetsky and Jones (2005): Classification of orientably regular maps of genus $p+1$ with 'large' automorphism groups (of order $> 6(g-1)$).

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Another consequence: A new proof of the classification result of Breda, Nedela, Š for regular maps on surfaces of genus $p+2$ for odd primes p.

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(b) If $p \equiv 1 \pmod 4$, then G is either one of the $(2j, 2l, 2)$-groups $G_{j,l}$ $\langle (x,y,z),\ r^{2j} = s^{2l} = (rs)^2 = (rs^{-1})^2 = 1\rangle \cong D_j \times D_l$ of order $4jl$, where $j \geq l \geq 3$, both j, l are odd, $(j,l) \leq 3$, $(j-1)(l-1) = 3p+1$, and $j \equiv l \not\equiv 1 \pmod 3$, or one of the $(6, 2l, 2)$-groups G_l with presentation $\langle (x,y,z),\ r^6 = s^{2l} = (rs)^2 = r^2s^2r^2s^{-2} = 1\rangle \cong (D_3 \times D_l).Z_3$ of order $36l$, where $l \equiv 2 \pmod 4$ and $2l - 3 = p$.

Directions of future research

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- $p = 7,\ G \cong \mathrm{PSL}(2, 13),\ |G| = 1092$, *with presentation*

$$\langle (x, y, z),\ r^{13} = s^3 = rs^{-1}r^2s^{-1}r^2sr^{-1}sr^{-1}z = r^{-5}s^{-1}r^5sr^{-4}sy = 1\rangle$$

MANY THANKS TO THE ORGANIZERS OF THIS NICE MEETING!