# Allpass Processes with Application to Finance

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## 1. Motivating Examples—MA(1)

Warm-up example:

Model 1: $X_t = Z_t - .5\, Z_{t-1}$, $\{Z_t\} \sim \text{IID}(0,\sigma^2)$

Model 2: $X_t = Z_t - 2\, Z_{t-1}$, $\{Z_t\} \sim \text{IID}(0,\sigma^2/4)$

Both models have the same autocovariance functions. Any difference between the two?

For model 2, there exists $\{\varepsilon_t\} \sim \text{WN}(0,\sigma^2)$, such that

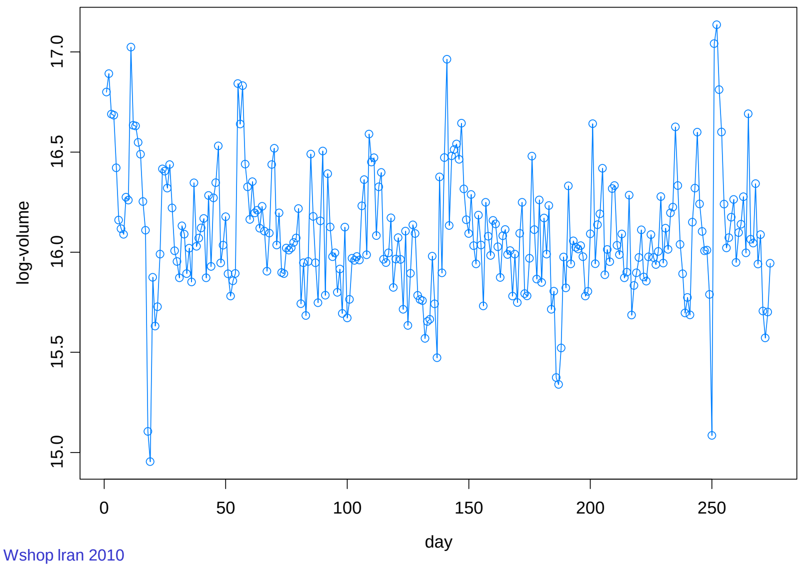
$$X_t = \varepsilon_t - .5\, \varepsilon_{t-1}, \quad \{\varepsilon_t\} \sim \text{WN}(0,\sigma^2).$$

But, $\{\varepsilon_t\}$ is not IID unless Guassian.

The sequence $\{\varepsilon_t\}$ is an example of an ***allpass*** process—*serially dependent without correlation*.

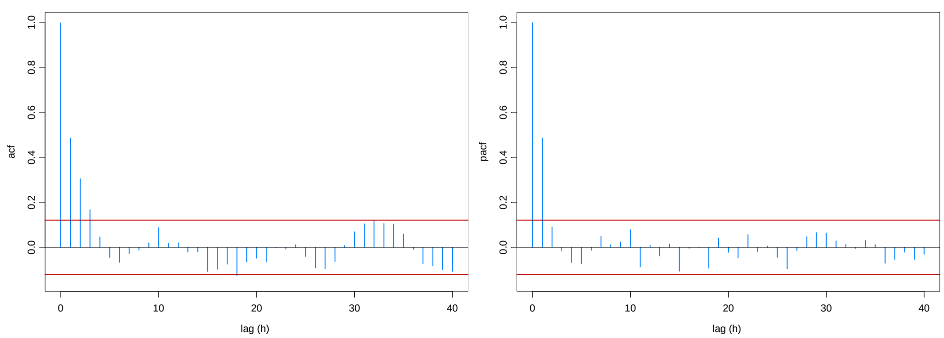
## 1. Motivating Example (cont)

Log(volume) of Walmart stock 12/1/03-12/31/04






## 1. Motivating Example (cont)




Analysis suggests that $\{X_t\}$ follows an AR (1) or AR(2).

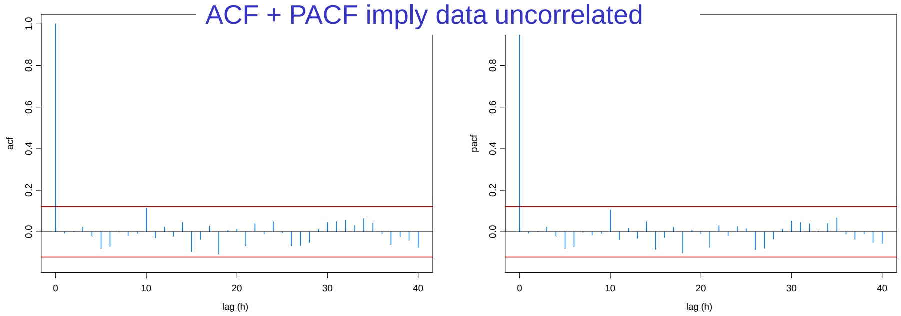
A *causal* AR(2) fit is

$$X_t = .4455\, X_{t-1} + .1025\, X_{t-2} + Z_t$$

Are the estimated residuals iid?

# Analysis of residuals from causal AR(2) fit

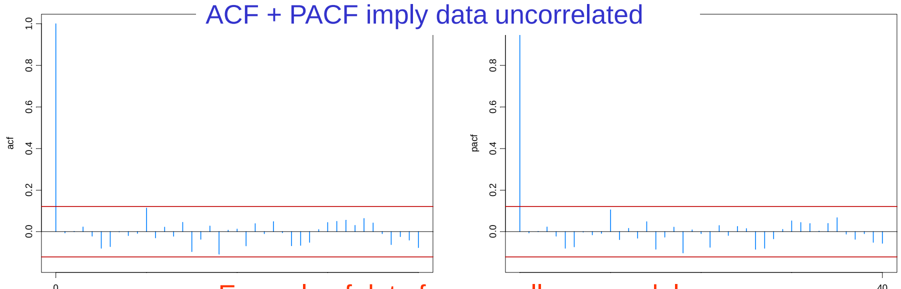
## ACF + PACF imply data uncorrelated



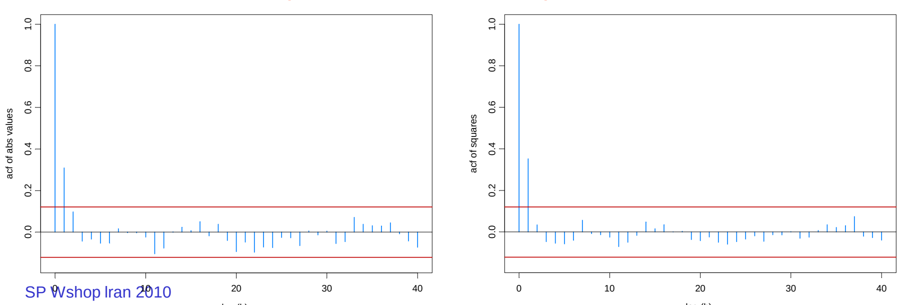



# Analysis of residuals from causal AR(2) fit

## ACF + PACF imply data uncorrelated




## Example of data from an allpass model



## Game Plan





## 2. Setup—simple allpass model

Allpass(1) model:

$$X_t - .5\, X_{t-1} = Z_t - 2\, Z_{t-1}, \quad \{Z_t\} \sim \text{IID}(0, \sigma^2)$$

Remarks:

1. Replace 2 by .5 on right-hand side and $\sigma^2$ by $4\sigma^2$ and get $X_t = Z_t$ (IID process)

2. $X_t$ is causal, i.e.,

$$X_t = \sum_{j=0}^{\infty} .5^j (Z_{t-j} - 2Z_{t-j-1}) = Z_t - 1.5\sum_{j=1}^{\infty} .5^{j-1} Z_{t-j} = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

3. $X_t$ is uncorrelated, WN(0, $4\sigma^2$).

$$\text{cov}(X_t, X_{t+h}) = 0, \quad h \neq 0$$

## 2. Setup—simple allpass model

Allpass(1) model:

$$X_t - .5\,X_{t-1} = Z_t - 2\,Z_{t-1}, \quad \{Z_t\} \sim \text{IID}(0, \sigma^2)$$

4. $X_t^2$ is correlated (as long as $Z_t$ is not Gaussian!).

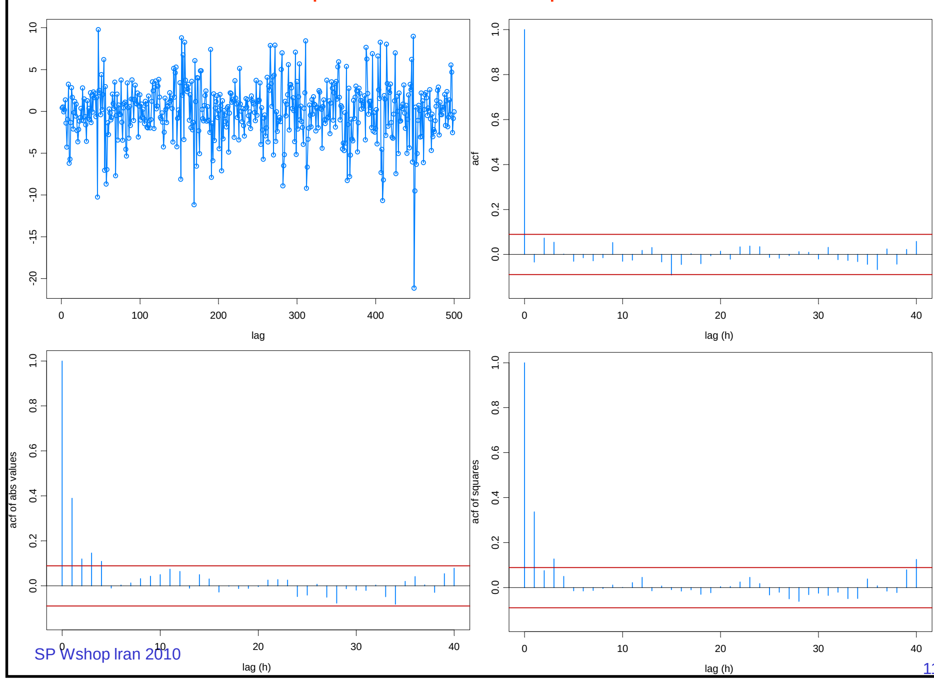
$$\text{cor}(X_t^2, X_{t+h}^2) = \frac{(EZ_t^4 - 3\sigma^4)\sum_{j=0}^{\infty}\psi_j^2\psi_{j+h}^2}{(EZ_t^4 - 3\sigma^4)\sum_{j=0}^{\infty}\psi_j^4 + 2\sigma^4\left(\sum_{j=0}^{\infty}\psi_j^2\right)^2} \neq 0 \quad \text{if} \quad EZ_t^4 \neq 3\sigma^4$$

$$= \frac{(EZ_t^4 - 3\sigma^4)12(.5)^{2h}}{(EZ_t^4 - 3\sigma^4)6.4 + 2\sigma^4 4}$$

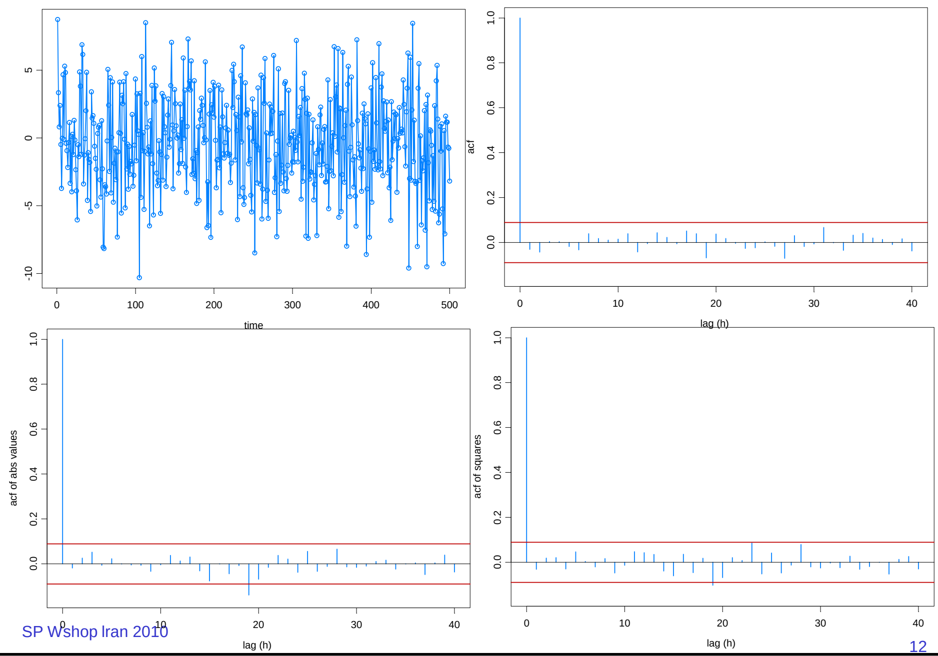
$$\approx 1.875(.5)^{2h}$$



## Sample realization of all-pass of order 1



## Sample realization of ***Gaussian*** all-pass of order 1






## 2. Setup—general allpass model

Causal AR polynomial: $\phi(z)=1-\phi_1 z - \cdots - \phi_p z^p$ , $\phi(z) \neq 0$ for $|z|\leq 1$.

Define MA polynomial:

$$\theta(z) = -z^p \phi(z^{-1})/\phi_p = -(z^p - \phi_1 z^{p-1} - \cdots - \phi_p)/\phi_p$$

$\neq 0$ for $|z|\geq 1$ (MA polynomial is non-invertible).

ARMA(p,p) model for data:

$$\{X_t\}: \ \phi(B)X_t = \theta(B) Z_t, \ \{Z_t\} \sim \text{IID (non-Gaussian)}$$

$$B^k X_t = X_{t-k}$$

Examples:

All-pass(1): $X_t - \phi X_{t-1} = Z_t - \phi^{-1} Z_{t-1}, \quad |\phi| < 1.$

All-pass(2): $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t + \phi_1/\phi_2 Z_{t-1} - 1/\phi_2 Z_{t-2}$

## 2. Setup—Allpass models

Properties: *linear process with nonlinear behavior*

- causal, non-invertible ARMA with MA representation

$$X_t = \frac{\theta(B)}{\phi(B)} Z_t = \frac{B^p \phi(B^{-1})}{-\phi_p \phi(B)} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

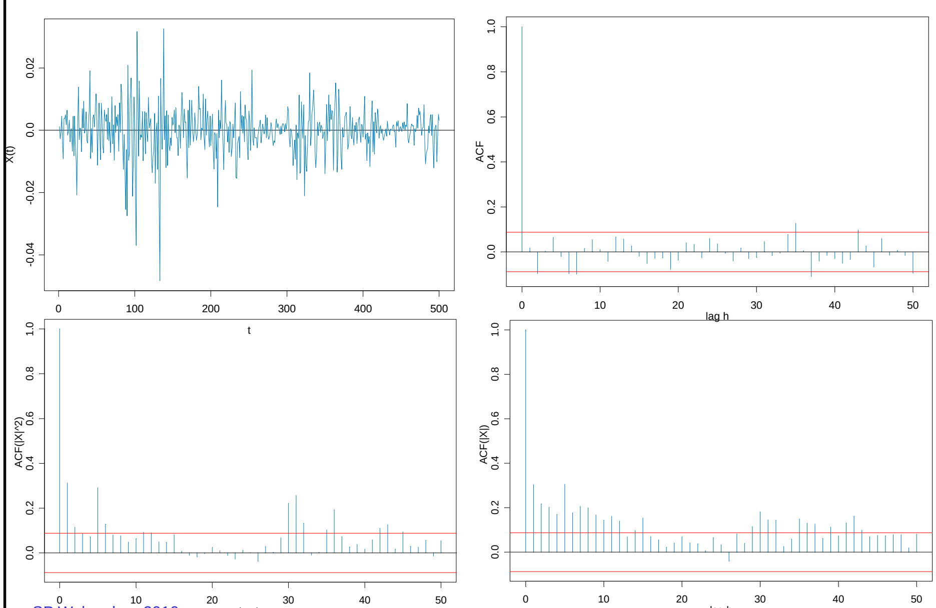
- uncorrelated (flat spectrum)

$$f_X(\omega) = \frac{|\theta(e^{i\omega})|^2}{|\phi(e^{-i\omega})|^2} \frac{\sigma^2}{2\pi} = \frac{|e^{-ip\omega}|^2 |\phi(e^{i\omega})|^2}{\phi_p^2 |\phi(e^{-i\omega})|^2} \frac{\sigma^2}{2\pi} = \frac{\sigma^2}{\phi_p^2 2\pi}$$

- zero mean
- data are dependent if and only if noise is non-Gaussian (e.g. Breidt & Davis 1991).
- squares and absolute values are correlated.
- $X_t$ is heavy-tailed if noise is heavy-tailed.



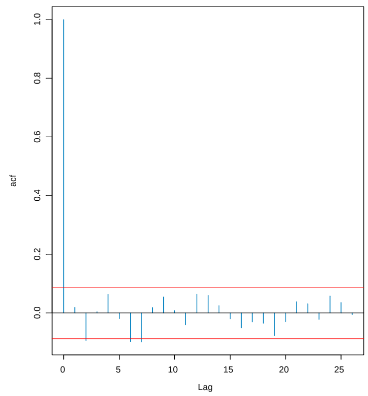
## 2. Setup—NZ/US exchange example (500 log-returns)



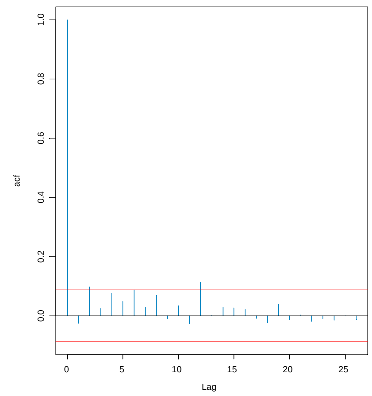
## 2. Setup— NZ-USA exchange example

Allpass model fitted to NZ-USA exchange rates :
Order = 6, $\phi_1$=.852, $\phi_2$=.616, $\phi_3$=. 952, $\phi_4$=.098, $\phi_5$= -.158, $\phi_6$=-.066

ACF: residuals




ACF: (residuals)$^2$






## 2. Setup—Allpass models

Application to noninvertible/noncausal modeling:

- Suppose the time series $X_t$ follows a noncausal AR and/or a non-invertible MA process. For definiteness, suppose $\{X_t\}$ follows the noninvertible MA model

$$X_t = \theta_i(B)\, \theta_{ni}(B)\, Z_t\,, \quad \{Z_t\} \sim \text{IID}.$$

Step 1: Let $\{U_t\}$ be the residuals obtained by fitting a purely invertible MA model, i.e.,

$$X_t = \hat{\theta}(B)U_t$$

$$\approx \theta_i(B)\tilde{\theta}_{ni}(B)U_t, \quad (\tilde{\theta}_{ni} \text{ is the invertible version of } \theta_{ni}).$$

So

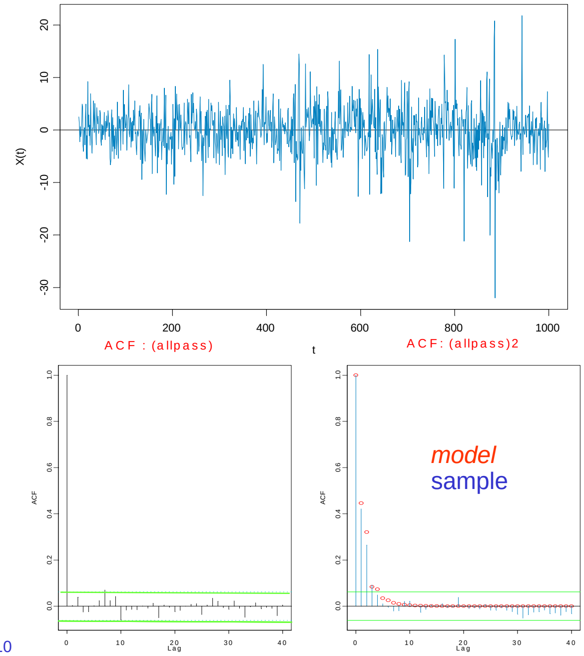
$$U_t \approx \frac{\theta_{ni}(B)}{\tilde{\theta}_{ni}(B)} Z_t \quad \text{or} \quad \tilde{\theta}_{ni}(B)U_t \approx \theta_{ni}(B)Z_t$$

Step 2: Fit a purely causal AP model to $\{U_t\}$

## 2. Allpass models

Realization from an all-pass model of order 2

(t3 noise )






## 3. Parameter Estimation

Estimation for All-Pass Models

- Second-order moment techniques do not work (work for causal AR!)
  - least squares
  - Gaussian likelihood
- Higher-order cumulant methods
  - Giannakis and Swami (1990)
  - Chi and Kung (1995)
- Non-Gaussian likelihood methods
  - likelihood approximation assuming known density
  - quasi-likelihood
- Other
  - LAD- least absolute deviation (Breidt, Davis, Trindade (2001))
  - Rank-based estimation (minimum dispersion) (Andrews, Davis, Breidt (2006,2007))

## 3. Approximating the likelihood

Data: $(X_1, \ldots, X_n)$

Model: $$X_t = \phi_{01} X_{t-1} + \cdots + \phi_{0p} X_{t-p} - (Z_{t-p} - \phi_{01} Z_{t-p+1} - \cdots - \phi_{0p} Z_t)/\phi_{0r}$$

where $\phi_{0r}$ is the last non-zero coefficient among the $\phi_{0j}$'s.

Noise: $$z_{t-p} = \phi_{01} z_{t-p+1} + \cdots + \phi_{0p} z_t - (X_t - \phi_{01} X_{t-1} - \cdots - \phi_{0p} X_{t-p}),$$

where $z_t = Z_t / \phi_{0r}$.

More generally define,

$$z_{t-p}(\phi) = \begin{cases} 0, & \text{if } t = n+p, \ldots, n+1, \\ \phi_1 z_{t-p+1}(\phi) + \cdots + \phi_p z_t(\phi) - \phi(B) X_t, & \text{if } t = n, \ldots, p+1. \end{cases}$$

Note: $z_t(\phi_0)$ is a close approximation to $z_t$ (initialization error)



Assume that $Z_t$ has pdf $f_\tau$ and consider the vector

$$\mathbf{z} = (\underbrace{X_{1-p}, \ldots, X_0, z_{1-p}(\phi), \ldots, z_0(\phi)}, \underbrace{z_1(\phi), \ldots, z_{n-p+1}(\phi), \ldots, z_n(\phi)})'$$

↖ independent pieces ↗

Joint density of $\mathbf{z}$:

$$h(\mathbf{z}) = h_1(X_{1-p}, \ldots, X_0, z_{1-p}(\phi), \ldots, z_0(\phi)) \bullet \left( \prod_{t=1}^{n-p} f_\tau(\phi_q z_t(\phi)) \, |\phi_q| \right) h_2(z_{n-p+1}(\phi), \ldots, z_n(\phi)),$$

and hence the joint density of the data can be approximated by

$$h(\mathbf{x}) = \left( \prod_{t=1}^{n-p} f_\tau(\phi_q z_t(\phi)) \, |\phi_q| \right)$$

where $q = \max\{0 \le j \le p : \phi_j \ne 0\}$.

Log-likelihood:

$$L(\phi,\sigma) = -(n-p)\ln(\sigma / |\phi_q|) + \sum_{t=1}^{n-p} \ln f(\sigma^{-1}\phi_q z_t(\phi))$$

where $f_\sigma(z) = \sigma^{-1} f(z/\sigma)$.

Least absolute deviations: choose Laplace density

$$f(z) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}\,|z|)$$

and log-likelihood becomes

$$\text{constant} - (n-p)\ln\kappa - \sum_{t=1}^{n-p} \sqrt{2}\,|z_t(\phi)|/\kappa, \quad \kappa = \sigma/|\phi_q|$$

Concentrated Laplacian likelihood

$$l(\phi) = \text{constant} - (n-p)\ln \sum_{t=1}^{n-p} |z_t(\phi)|$$

Maximizing $l(\phi)$ is equivalent to minimizing the absolute deviations

$$m_n(\phi) = \sum_{t=1}^{n-p} |z_t(\phi)|.$$



Rank-based Estimation:

Minimize the objective function

$$S(\phi) = \sum_{t=1}^{n-p} \varphi\left(\frac{t}{n-p+1}\right) z_{(t)}(\phi)$$

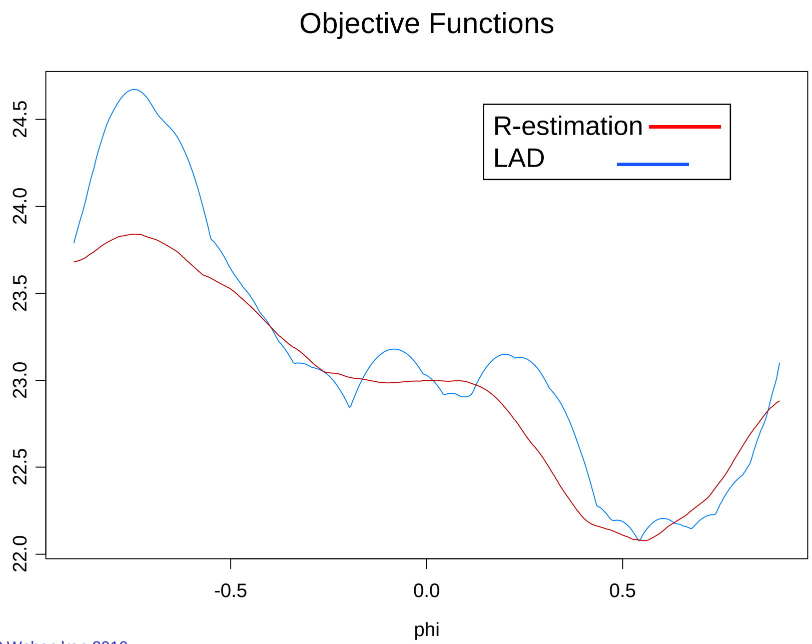
where $\{z_{(t)}(\phi)\}$ are the ordered $\{z_t(\phi)\}$, and the weight function $\varphi$ satisfies:

- $\varphi$ is differentiable and nondecreasing on (0,1)
- $\varphi'$ is uniformly continuous
- $\varphi(x) = -\varphi(1-x)$

Remark: For LAD, take

$$\varphi(x) = \begin{cases} -1, & 0 < x < 1/2, \\ 1, & 1/2 < x < 1. \end{cases}$$

The R-estimation objective function is smoother than the LAD-objective function and hence easier to minimize.

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## Limit Results

- For LAD estimators of AP(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{LAD}} - \phi_0) \xrightarrow{D} N\left(0, \frac{\mathrm{Var}(|Z_1|)}{2(2\sigma^2 f_\sigma(0) - E|Z_1|)^2}\sigma^2\Gamma_p^{-1}\right)$$

- For MLE estimators of AP(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{MLE}} - \phi_0) \xrightarrow{D} N\left(0, \frac{1}{2(\sigma^2 \hat{I} - 1)}\sigma^2\Gamma_p^{-1}\right)$$

- For Rank-based estimators of AP(p):

$$\sqrt{n}(\hat{\phi}_{\mathrm{R}} - \phi_0) \xrightarrow{D} N\left(0, \frac{\sigma^2 \tilde{J} - \tilde{K}^2}{2(\sigma^2 \tilde{L} - \tilde{K})^2}\sigma^2\Gamma_p^{-1}\right)$$

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Laplace: (LAD=MLE)

R: $\frac{\sigma^2 \tilde{J} - \tilde{K}^2}{2(\sigma^2 \tilde{L} - \tilde{K})^2} = \frac{5}{6}$ (using $\varphi(x) = x - 1/2$, Wilcoxon)

LAD=MLE: $\frac{\mathrm{Var}(|Z_1|)}{2(2\sigma^2 f_\sigma(0) - E|Z_1|)^2} = \frac{1}{2(\sigma^2 \hat{I} - 1)} = \frac{1}{2}$

Students $t_\nu$:

| ν | LAD | R | MLE | LAD/R | MLE/R |
|---|---|---|---|---|---|
| 3 | .733 | **.520** | .500 | **1.411** | .962 |
| 6 | 6.22 | **3.01** | 3.00 | **2.068** | .997 |
| 9 | 16.8 | **7.15** | 7.00 | **2.354** | .980 |
| 12 | 32.6 | **13.0** | 12.5 | **2.510** | .964 |
| 15 | 53.4 | **20.5** | 19.5 | **2.607** | .952 |
| 20 | 99.6 | **36.8** | 34.5 | **2.707** | .937 |
| 30 | 234 | **83.6** | 77.0 | **2.810** | .921 |

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## 3. Stable noise

The noise is assumed to have a nonGaussian stable distribution

Noise distribution: $\{Z_t\}$ ~ IID with a stable distribution

That is, $Z_t$ has characteristic function given by

$$\varphi_0(s) = E\{\exp(isZ_t)\} = \exp\{-\sigma^\alpha |s|^\alpha [1 + i\beta \,\mathrm{sgn}(s) \tan(\pi\alpha/2)(|\sigma s|^{1-\alpha} - 1)] + i\mu s\}$$

where

+ exponent $\alpha \in (0,2)$

+ symmetry parameter $\beta \in (-1,1)$

+ scale parameter $\sigma > 0$

+ location parameter $\mu$.

Density function: inverse Fourier transform

$$f(z;(\alpha,\beta,\sigma,\mu)) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-isz} \varphi_0(s)\, ds$$

can be computed numerically reasonably quickly (Nolan, `97).

## 3. Stable noise(cont)

Remarks:

1. There are a number of parameterizations of a stable distribution (see Zolotarev `86), but we use the one advocated by Nolan `01.
   - differentiable wrt to z
   - differentiable wrt to the parameter vector $(\alpha,\beta,\sigma,\mu)'$
2. Stable distributions have heavy (Pareto-like) tails

$$x^{\alpha} P(|Z_1| > x) \to \sigma^{\alpha} c(\alpha) \quad \text{and} \quad c(\alpha) = \left( \int_0^{\infty} t^{-\alpha} \sin(t) dt \right)^{-1}.$$

3. Location/scale family

$$f\left(z;(\alpha,\beta,\sigma,\mu)\right) = \sigma^{-1} f\left(\sigma^{-1}(z-\mu);(\alpha,\beta,1,0)\right)$$

4. Unimodal. $f\left(\bullet;(\alpha,\beta,1,0)\right)$ is unimodal



## 3. MLE—allpass models

MLE based on stable noise:

$$n^{1/\alpha_0}(\hat{\phi}_{ML} - \phi_0) \to_d \xi, \qquad n^{1/2}(\hat{\tau}_{ML} - \tau_0) \to_d \eta \sim \mathrm{N}(0, \mathrm{I}^{-1}(\tau_0)).$$

Remarks:

- The distribution of ξ is generally intractable. However, one can use bootstrapping techniques (see later example and Davis and Wu `97).
- The scaling for the AR parameters is $n^{1/\alpha}$, which is must faster than the standard $n^{1/2}$ rate.
- The limit behavior of the estimates of the stable parameters is the same as for an iid sequence (see DuMouchel `73).

## 3. MLE—allpass models

Idea behind proofs (stable case).

Reparameterize: Denote the true parameters by $\phi_0$ and $\tau_0$ and set

$$u = n^{1/\alpha_0}(\phi - \phi_0) \quad \text{and} \quad v = n^{1/2}(\tau - \tau_0)$$

Now the log-likelihood can be re-expressed as a continuous function on $R^p \times R^4$ given by

$$\begin{aligned} W_n(u,v) &= L(\phi_0 + n^{-1/\alpha_0}u, \tau_0 + n^{-1/2}v) - L(\phi_0, \tau_0) \\ &= \sum_{t=1}^{n-p} \ln\left[f\left(z_t(\phi_0 + n^{-1/\alpha_0}u); \tau_0 + n^{-1/2}v\right)\right] - \sum_{t=1}^{n-p} \ln\left[f\left(z_t(\phi_0); \tau_0\right)\right] \end{aligned}$$

Note: $(\hat{u}_n, \hat{v}_n) = \arg\max_{u,v} W_n(u,v) = \left(n^{1/\alpha_0}(\hat{\phi} - \phi_0), n^{1/2}(\hat{\tau} - \tau_0)\right)$



Result: $W_n(u,v) \to_d W(u) + v'\mathbf{N} - 2^{-1}v'\mathbf{I}(\tau_0)v$ on $C(R^p \times R^4)$, where

- $\mathbf{I}(\tau_0) := -\left[\mathrm{E}\{\partial^2 \ln f(z;\tau)/(\partial\tau_i \partial\tau_j)\}\right]_{i,j=1}^4$ is the Fisher information for a stable density.
- $\mathbf{N} \sim \mathrm{N}(\mathbf{0}, \mathbf{I}(\tau_0))$ is a normal random vector independent of W.
- W is the process

$$W(u) = \sum_{k=1}^{\infty} \sum_{j \neq 0} \left\{ \ln f\left(z_{k,j} + [\tilde{c}(\alpha_0)]^{1/\alpha_0} \sigma_0 \,|\phi_{0r}|^{-1} c_j(u) \delta_k \Gamma_k^{-1/\alpha_0}; \tau_0\right) - \ln f\left(z_{k,j}; \tau_0\right) \right\}$$

  - $c_j(u)$ are found through the Laurent series identity

$$\sum_{j \neq 0} c_j(u) z^j = u'\left[-(z^{-k}/\phi_0(z)) + (z^k/\phi_0(z^{-1}))\right]_{k=1}^p$$

  - $\{z_{k,j}\}$ is iid $z_{1,1} =_d Z_1/\phi_{0r}$
  - $\{\delta_k\}$ is iid $P(\delta_k=1) = p = 1 - P(\delta_k=-1)$.
  - $\Gamma_k = E_1 + \cdots + E_k$, where $\{E_k\}$ is iid unit exponentials
  - $\{z_{k,j}\}$, $\{\delta_k\}$, and $\{E_k\}$ are mutually independent

Deconstructing the limit result and limit process:

The limit process in the following

$$W_n(u,v) \to_d W(u) + v'\mathbf{N} - 2^{-1}v'\mathbf{I}(\tau_0)v$$

consists of two independent pieces:

1. W(u) which governs the limit behavior of the AP model parameters $\phi_1,\ldots,\phi_p$. Then $\xi = \arg\max_u W(u)$ is the limit rv for $\hat{\phi}_{ML}$.
2. $v'\mathbf{N} - 2^{-1}v'\mathbf{I}(\tau_0)v$ which governs the limit behavior of the stable parameters $(\alpha,\beta,\sigma,\mu)$. Set $\eta = \arg\max_v v'\mathbf{N} - 2^{-1}v'\mathbf{I}(\tau_0)v$, is equal to
   $$\eta = \mathbf{I}^{-1}(\tau_0)\mathbf{N} \sim \mathrm{N}(0, \mathbf{I}^{-1}(\tau_0)).$$

For standard MLE (or LAD and Rank): relevant SP convergent result

$$W_n(u,v) \to_d u'\mathbf{N}_1 - 2^{-1}u'c\boldsymbol{\Gamma}_p u + v'\mathbf{N}_2 - 2^{-1}v'\mathbf{I}(\tau_0)v$$

where $\mathbf{N}_1 \sim \mathrm{N}(\mathbf{0}, \Gamma_p)$ and $\mathbf{N}_2 \sim \mathrm{N}(\mathbf{0}, \mathbf{I}(\tau_0))$, are independent. Then

$$n^{1/2}(\hat{\phi}_{ML} - \phi_0) \to_d \xi \quad \text{and} \quad n^{1/2}(\hat{\tau}_{ML} - \tau_0) \to_d \eta.$$



## 4. AR models—causal and noncausal

Model:

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$$

$$(1-\phi_1 B - \cdots - \phi_p B^p)\, X_t = Z_t \quad \text{B=backward shift operator}$$

$$(1-\phi_1 B - \cdots - \phi_r B^r)\,(1-\phi_{r+1} B - \cdots - \phi_{r+s} B^s)\, X_t = Z_t$$

$$\phi^+(B)\phi^*(B)\, X_t = Z_t$$

where $\phi^+(z)$ is the good (***causal***) AR polynomial and $\phi^*(B)$ is the bad (***purely noncausal***) AR polynomial.

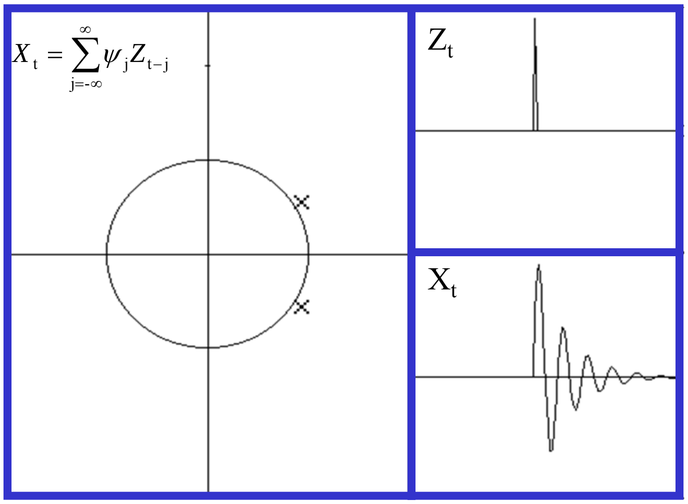
- $\phi^+(z)$ ***causal*** means $\phi^+(z) \neq 0$ for $|z| \le 1$.
- $\phi^*(z)$ ***purely noncausal*** means $\phi^*(z) \neq 0$ for $|z| > 1$

The process has the two-sided representation (one-sided if r or s = 0).

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t+j}$$

# 4. AR models—causal and noncausal

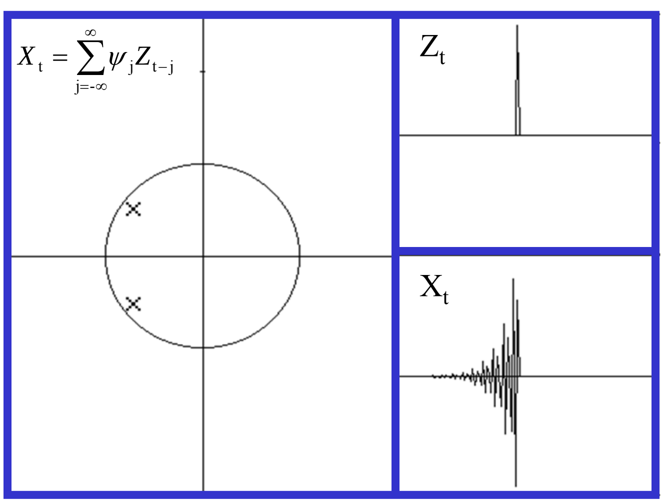
Impulse response: causal & low frequency





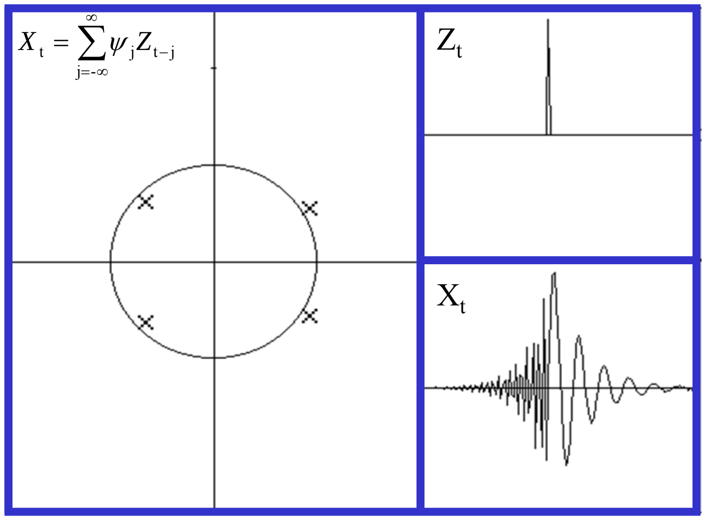

# 4. AR models—causal and noncausal

Impulse response: noncausal & high frequency



## 4. AR models—causal and noncausal

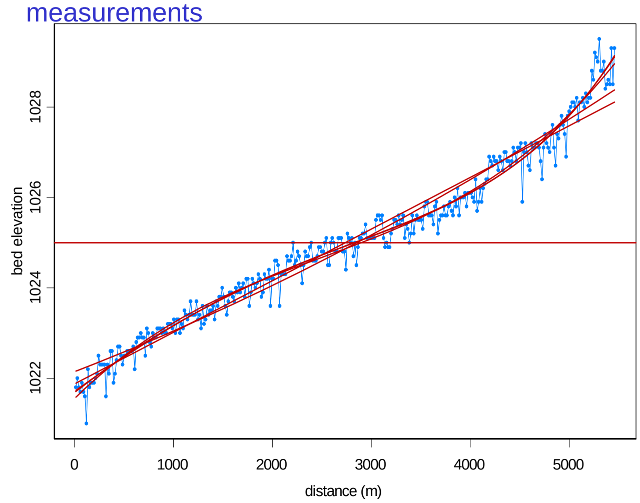
Impulse response: mixed causal (low frequency) & noncausal (high frequency)





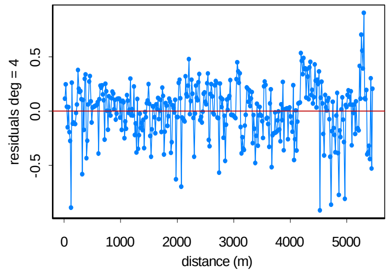

## Muddy Creek- tributary to Sun River in Central Montana

Muddy Creek: surveyed every 15.24 meters, total of 5456m; 358 measurements



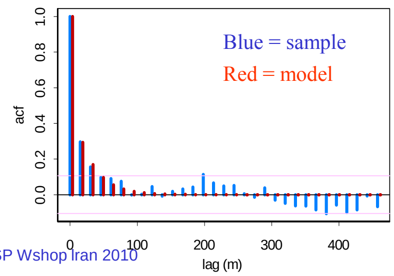

| Degree | $AIC_c$ | ARMA |
|---|---|---|
| 0 | 59.67 | (1,2) |
| 1 | 26.98 | (2,1) |
| 2 | 26.30 | (1,1) |
| 3 | 7.12 | (1,1) |
| 4 | 2.78 | (1,1) |
| 5 | 4.68 | (1,1) |
| 4 | 2.78 | (1,1) |

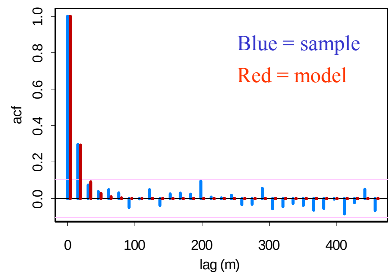
# Muddy Creek: residuals from poly(d=4) fit




Minimum AIC$_c$ ARMA model:
ARMA(1,1)
$Y_t = .574\, Y_{t\text{-}1} + Z_t - .311\, Z_{t\text{-}1}$,
$\{Z_t\}$~WN(0,.0564)

Residuals follow allpass model AP(1)$\Rightarrow$
*Noncausal ARMA(1,1) model*:
$Y_t = 1.743\, Y_{t\text{-}1} + \varepsilon_t - .311\, \varepsilon_{t\text{-}1}$ , $\{\varepsilon_t\}$~IID




SP Wshop Iran 2010

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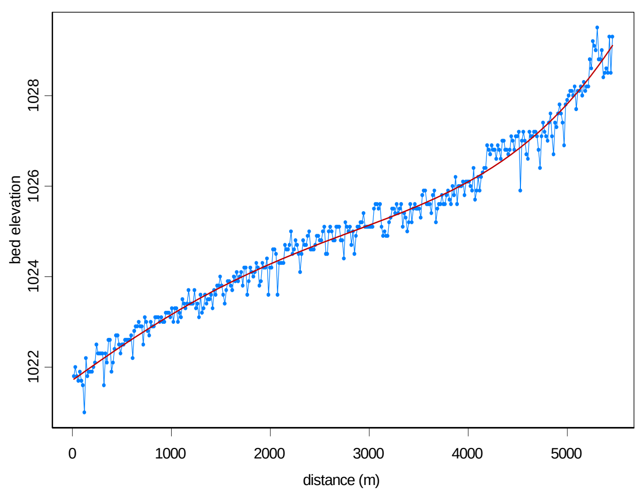


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## 5. Simulation Results

Simulation setup:

- 300 replicates of a non-causal AR(2) model

  $X_t = -1.2\, X_{t-1} + 1.6\, X_{t-2} + Z_t$ (zeros of AR polyn 1.25, -.5)

- noise distribution is stable with two sets of parameter values:
  - $\alpha = .8 \quad \beta = .5 \quad \sigma = 1.0 \quad \mu = 0.0$ (really heavy!)
  - $\alpha = 1.5 \quad \beta = .5 \quad \sigma = 1.0 \quad \mu = 0.0$
- sample size n=500
- estimation is maximum likelihood over non-normal stable.



## 5. Simulation Results

| | Asymp SD | Empirical Mean | Empirical Std Dev | | Asymp SD | Empirical Mean | Empirical Std Dev |
|---|---|---|---|---|---|---|---|
| $\phi_1 = -1.2$ | | -1.200 | 0.004 | $\phi_1 = -1.2$ | | -1.200 | 0.004 |
| $\phi_2 = 1.6$ | | 1.600 | 0.004 | $\phi_2 = 1.6$ | | 1.600 | 0.004 |
| $\alpha = 0.8$ | 0.051 | 0.798 | 0.041 | $\alpha = 0.8$ | 0.049 | 0.800 | 0.039 |
| $\beta = 0.0$ | 0.067 | -0.001 | 0.068 | $\beta = 0.5$ | 0.058 | 0.502 | 0.056 |
| $\sigma = 1.0$ | 0.077 | 0.997 | 0.073 | $\sigma = 1.0$ | 0.074 | 0.997 | 0.071 |
| $\mu = 0.0$ | 0.054 | -0.002 | 0.057 | $\mu = 0.0$ | 0.062 | -0.004 | 0.064 |
| $\phi_1 = -1.2$ | | -1.212 | 0.083 | $\phi_1 = -1.2$ | | -1.204 | 0.078 |
| $\phi_2 = 1.6$ | | 1.605 | 0.065 | $\phi_2 = 1.6$ | | 1.598 | 0.062 |
| $\alpha = 1.5$ | 0.071 | 1.502 | 0.069 | $\alpha = 1.5$ | 0.070 | 1.499 | 0.071 |
| $\beta = 0.0$ | 0.137 | 0.010 | 0.128 | $\beta = 0.5$ | 0.121 | 0.509 | 0.128 |
| $\sigma = 1.0$ | 0.048 | 0.999 | 0.066 | $\sigma = 1.0$ | 0.047 | 0.997 | 0.056 |
| $\mu = 0.0$ | 0.078 | -0.006 | 0.078 | $\mu = 0.0$ | 0.078 | 0.000 | 0.083 |

## 6. Walmart revisited---residuals from noncausal model

Analysis of the ACF and PACF of the time series (n=274) suggests that $\{X_t\}$ follows an AR (1) or AR(2).

A *causal* AR(2) fit (using Gaussian MLE) is

$$X_t = .4455\, X_{t-1} + .1025\, X_{t-2} + Z_t$$

The estimated residuals were uncorrelated but ***dependent***.



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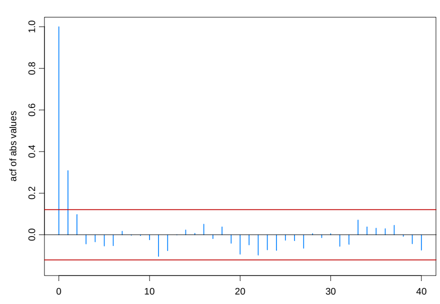


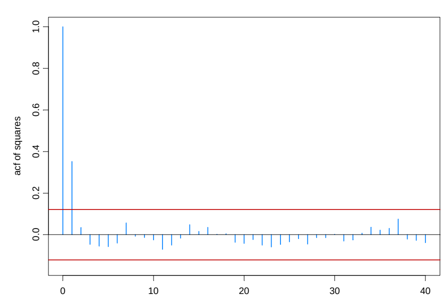

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Maximum-likelihood model:

$$X_t = -2.0766\, X_{t-1} + 2.0772\, X_{t-2} + Z_t \quad \text{(purely non-causal)}$$

$\{Z_t\}$ ~ IID Stable

$\alpha = 1.8335,\ \beta = .5650,\ \sigma = .4559,\ \mu = 16.0030$

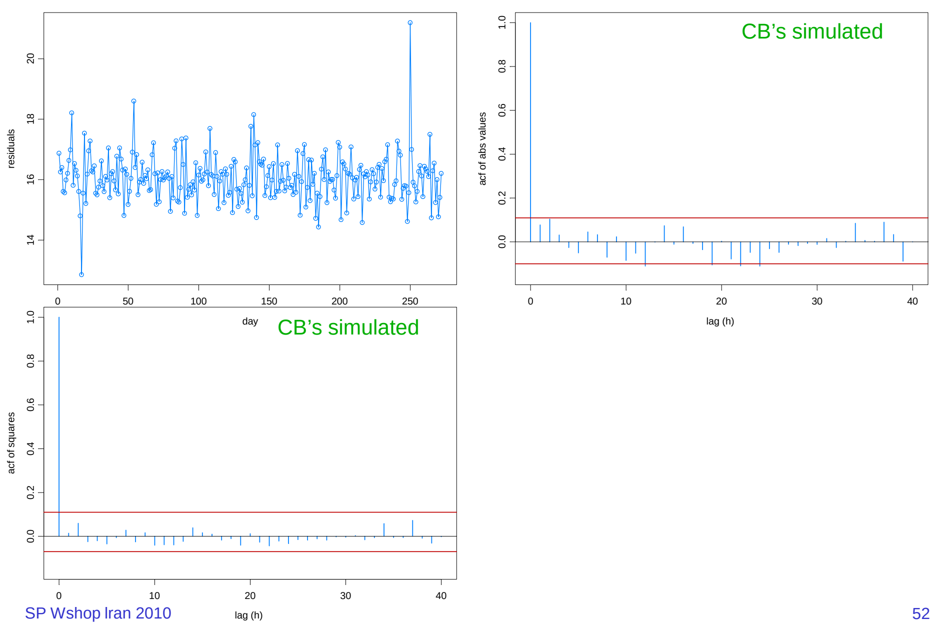
Bootstrap confidence intervals for $\phi_1$ and $\phi_2$ : (based on Davis and Wu `97)

(-2.5719,-1.4212)      (1.4521,2.5743)

Resample size is m=135.



## 6. Walmart—analysis of residuals from noncausal model



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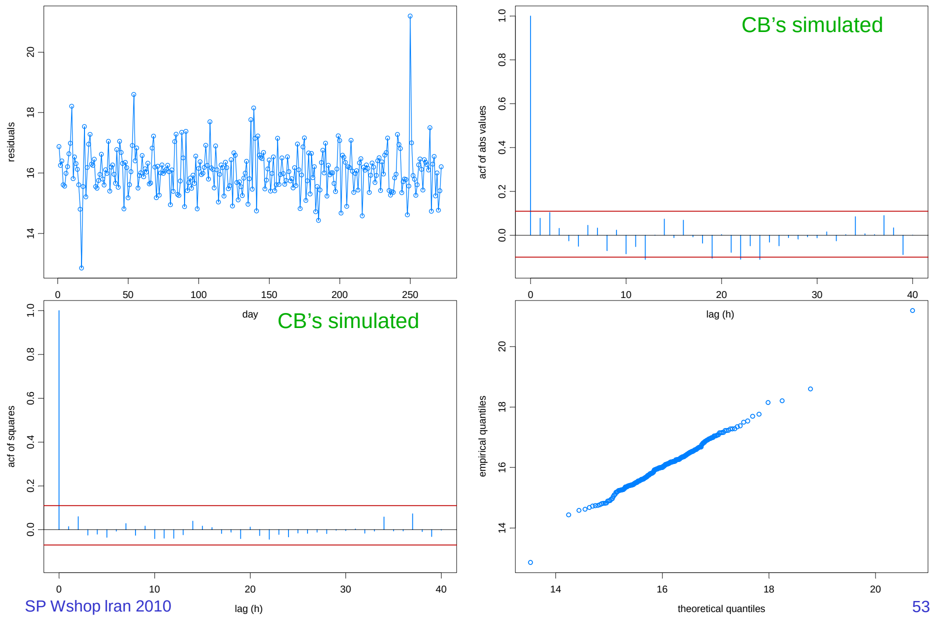




# Wrap up

1. Allpass models are linear time series models with nonlinear behavior (i.e., uncorrelated but dependent).

2. Allpass models are useful for identifying non-causal and noninvertible ARMA models.

3. Noncausal and noninvertible models may be more prevalent than one may have thought—certainly useful in spatial contexts and are being looked at in economics, fiscal foresight.

4. A non-causal AR model with non-normal stable noise appears to be a good fit for the log(volume) of Walmart stock.

5. MLE for $\alpha$-stable AR and AP models is tractable.

6. While limit theory is complicated, convergence rates are fast. Bootstrapping is a viable alternative for approximating the sampling distribution.

7. AP modeling should become part of standard TS modeling toolbox.