EXTENDED ABSTRACT

Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables and M a finitely generated $\mathbb{Z}^n$-graded S-module. Let $m \in M_a$ be a homogeneous element and Z a subset of $\{x_1, \ldots, x_n\}$. We denote by $mK[Z]$ the K-subspace of M whose basis consists of all homogeneous elements of the form mv where v is a monomial in $K[Z]$. The K-subspace $mK[Z] \subset S$ is called a *Stanley space* of dimension $|Z|$ if $mK[Z]$ is a free $K[Z]$-module.

A decomposition $\mathscr{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimal dimension of a Stanley space in the decomposition $\mathscr{D}$ is called the *Stanley depth* of $\mathscr{D}$, denoted $\operatorname{sdepth}(\mathscr{D})$.

We set $\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathscr{D})\colon \mathscr{D} \text{ is a Stanley decomposition of } M\}$, and call this number the *Stanley depth* of M. In [5, Conjecture 5.1] Stanley conjectured the inequality $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$.

In this talk M is always a finitely generated $\mathbb{Z}^n$ graded S-module with the property that $\dim_K(M_a) \leq 1$ for each $a \in \mathbb{Z}^n$. We will show that the Stanley depth of M can be determined in a finite number of steps. This is an easy generalization of the same result by Herzog, Vladoiu and Zheng [2] for $\mathbb{Z}^n$-graded modules of the form I/J where $J \subset I$ are monomial ideals in S.

We define a natural partial order on $\mathbb{Z}^n$ as follows: $a \leq b$ if and only if $a(i) \leq b(i)$ for $i = 1, \ldots, n$. Observe that $\mathbb{Z}^n$ with the partial order introduced is a distributive lattice with meet $a \wedge b$ and join $a \vee b$ defined as follows: $(a \wedge b)(i) = \min\{a(i), b(i)\}$ and $(a \vee b)(i) = \max\{a(i), b(i)\}$. We also denote by ε_j the jth canonical unit vector in $\mathbb{Z}^n$.

Suppose $m_1, \ldots, m_t$ is a minimal set of generators of M with $\deg m_i = a_i$. We choose $g \in \mathbb{Z}^n$ such that $a_i \leq g$ for all i , and let P_M^g be the set of all $c \in \mathbb{Z}^n$ with $c \leq g$ and such that $a_i \leq c$ for some i. The set P_M^g viewed as a subposet of $\mathbb{Z}^n$ is a finite poset. This poset is called the *characteristic poset* of M with respect to g. There is a natural choice for g, namely the join of all the a_i. For this g, the poset P_M^g has the least number of elements, and we denote it simply by P_M. Note that if Δ is a simplicial complex on the vertex set $[n]$, then P_{S/I_Δ} is just the face poset of Δ.

Remark 0.1. Notice that for each $a \in P_M^g$ there is a unique homogenous element $m \in M$ such that $\deg m = a$. We denote this unique element by m_a.

Let P be a poset and $a, b \in P$. We set $[a, b] = \{c \in P\colon \ a \leq c \leq b\}$ and call $[a, b]$ an *interval*. Of course, $[a, b] \neq \emptyset$ if and only if $a \leq b$. Suppose P is a finite poset. A *partition* of P is a disjoint union

$$\mathscr{P}\colon \ P = \bigcup_{i=1}^{r} [a_i, b_i]$$

of intervals.

We show that each partition of P_M^g gives rise to a Stanley decomposition of M. In order to describe the Stanley decomposition of M coming from a partition of P_M^g we need the following notation: for each $b \in P_M^g$, we set $Z_b = \{x_j\colon \ b(j) = g(j)\}$. We also introduce the function

$$\rho\colon \ P_M^g \to \mathbb{Z}_{\geq 0}, \quad c \mapsto \rho(c),$$

where $\rho(c) = |\{j\colon \ c(j) = g(j)\}| = |Z_c|$.

The following theorem was proved in [2] for the case that M is an $\mathbb{N}^n$-graded S module of the form J/I, where I and J are monomial ideals.

Theorem 0.2.

(a) *Let* $\mathscr{P}\colon P_M^g=\bigcup_{i=1}^r[c_i,d_i]$ *be a partition of* P_M^g. *Then*

$$\mathscr{D}(\mathscr{P})\colon M=\bigoplus_{i=1}^{r}(\bigoplus_{c} m_cK[Z_{d_i}])$$

is a Stanley decomposition of M, *where* m_c *is the unique homogenous element in* P_M^g *of degree* c *and the inner direct sum is taken over all* $c\in[c_i,d_i]$ *for which* $c(j)=c_i(j)$ *for all* j *with* $x_j\in Z_{d_i}$. *Moreover,* $\operatorname{sdepth}\mathscr{D}(\mathscr{P})=\min\{\rho(d_i)\colon\ i=1,\dots,r\}$.

(b) *Let* $\mathscr{D}$ *be a Stanley decomposition of* M. *Then, there exists a partition* $\mathscr{P}$ *of* P_M^g *such that*

$$\operatorname{sdepth}\mathscr{D}(\mathscr{P})\geq\operatorname{sdepth}\mathscr{D}.$$

As an immediate consequence we have the following:

Corollary 0.3. *Let* M *be a finitely generated* $\mathbb{Z}^n$ *graded* S*-module. Then*

$$\operatorname{sdepth}M=\max\{\operatorname{sdepth}\mathscr{D}(\mathscr{P})\colon \mathscr{P}\quad \textit{is a partition of}\quad P_M^g\}.$$

In particular, there exists a partition $\mathscr{P}$: $P_M^g=\cup_{i=1}^t[c_i,d_i]$ *of* P_M^g *such that*

$$\operatorname{sdepth}M=\ min\{\rho(d_i)\colon i=1,\cdots,t\}.$$

Let M be a finitely generated $\mathbb{Z}^n$ graded S-module. A chain $\mathscr{F}\colon (0)=M_0\subset M_1\subset\cdots\subset M_r=M$ of submodules of M with the property $M_i/M_{i-1}\cong S/P_i(-a_i)$ for $i=1,\cdots,r$ is called a prime filtration of M. It is was shown in [3] that if $\mathscr{F}$ is a prime filtration of M, then

$$\mathscr{D}(\mathscr{F})=\bigoplus_{i=1}^{r}m_{a_i}K[Z_i],$$

is a Stanley decomposition of M, where $Z_i=\{x_j\colon x_j\notin P_i\}$ and m_{a_i} is a homogenous element in M of degree a_i with the property that $\operatorname{Ann}_S(m_{a_i})=P_i$. It is easy to see that if $\mathscr{F}$ is a prime filtration of M, then

$$\dim(M)=\max\{\dim(S/P_i)\colon i=1,\cdots,r\}.$$

In [4] the author classified all Stanley decomposition $\mathscr{D}$ of S/I which is induced by a prime a filtration. In [2] the same result was shown for $\mathbb{Z}^n$ graded S- modules.

Proposition 0.4. [2] *Let* M *be a finitely generated* $\mathbb{Z}^n$*-graded* S*-module and* $\mathscr{D}: M=\bigoplus_{i=1}^m u_iK[Z_i]$ *a Stanley decomposition of* M. *Then the following conditions are equivalent:*

(a) $\mathscr{D}$ *is induced by a prime filtration.*

(b) *After a suitable relabeling of the summands in* $\mathscr{D}$ *we have* $M_j=\bigoplus_{i=1}^j u_iK[Z_i]$ *is a* $\mathbb{Z}^n$*-graded submodule of* M *for* $j=1,\dots,m$.

We shall need the following:

Theorem 0.5. *Let* M *be a* $\mathbb{Z}^n$*-graded* S *module, and Let* $\mathscr{P}\colon P_M^g=\bigcup_{i=1}^r[c_i,d_i]$ *be a partition of* P_M^g *with the property that for all* j *the union* $\bigcup_{i=1}^j[c_i,d_i]$ *is a poset ideal of* P_M^g. *Then* $\mathscr{D}(\mathscr{P})$ *is induced by a prime filtration.*

Theorem 0.5 can be used to compute the Krull dimension of M.

Corollary 0.6. *Let* M *be a* $\mathbb{Z}^n$*-graded* S *module. Then* $\dim M=\max\{\rho(c)\colon\ c\in P_M^g\}$.

REFERENCES

[1] J. Herzog, A. Soleyman Jahan, X. Zheng, Skeltons of monomial ideals, to apear in Mathematische Nachriechten.

[2] J. Herzog, M. Vladoiu, X. Zheng, How to compute the Stanley depth of a monomial ideal, J. Alg.

[3] A. Soleyman-Jahan, Stanley decompositions of squarefree modules and Alexander duality, Manuscripat mathematica,

[4] A. Soleyman Jahan, Prime filtrations of monomial ideals and polarizations. J. Algebra, **312** (2007), 1011–1032.

[5] R. P. Stanley, Linear Diophantine equations and local cohomology, Invent. Math. **68**, (1982), 175–193.

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