THE HILBERT FUNCTION OF A SET OF 2-FAT POINTS IN MULTI-PROJECTIVE SPACE

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Abstract. Let k be an algebraically closed field and $V_1, \ldots, V_t$ be k-vector spaces of dimensions $n_1 + 1, \ldots, n_t + 1$, respectively, and $V = V_1 \otimes \cdots \otimes V_t$ be the tensor product of these vector spaces. An element of V is called decomposable if it can be represented as $v_1 \otimes \cdots \otimes v_t$. It is clear every element of V can be represented as a linear combination of decomposable tensors. For a given element $v \in V$ the smallest integer s which v can be represented as s decomposable tensors is called the tensor rank of v.

For a given positive integer s the set of element of V which are limit of rank s, form a projective variety and a natural question is:
What is the dimension of this variety?

For $s = 1$ this variety is equal to X, the Segre embedding of $\mathbb{P}^{n_1} \otimes \cdots \otimes \mathbb{P}^{n_t}$ into $\mathbb{P}^N$, where $N = \sum_1^t (n_i + 1) - 1$, and for $s > 1$ this variety is the s^{th} secant variety of X.

If we switch to dual vector spaces, then the above problem can be restated as follows:
Let $R = k[x_{01}, x_{11}, \ldots, x_{n_1 1}, x_{02}, \ldots, x_{n_t t}]$ and F be a homogeneous polynomial of degree t in R. Then for the given positive integer s is it possible to represent F as

$$L_{11}L_{12} \cdots\ L_{1t} + \cdots + L_{s1}L_{s2} \cdots L_{st},$$

where L_{ij}'s are linear forms in $k[x_{0i}, \ldots, x_{n_i i}]$.
If dimension of the s^{th} secant variety of X is N then the answer to this question would be positive.

This question is known as Waring problem's for multi-homogeneous polynomials and is widely open, which except for special cases, the complete answer is not available.

An approach to find a satisfiable answer to this problem is to compute the value of the Hilbert function of a scheme of s generic double points in $\mathbb{P}^{n_1} \otimes \cdots \otimes \mathbb{P}^{n_t}$. Whenever these points are coordinate points of $\mathbb{P}^{n_1} \otimes \cdots \otimes \mathbb{P}^{n_t}$, a nice combinatorial method can be used to compute this dimension.

In this talk we describe the problem in this case and illustrate how this combinatorial method provides an answer to this special case.