Positive Self-similar Markov Processes

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- There is a Lamperti transformation T_1 is between self- similar processes (ssp) and stationary processes (stp).
- the second one T_2 or T is betwen positive self similar Markov processes (pssMp) and Lévy processes (Lp).(the main interest of this talk)
- the third one T_3 is between spectrally postive Lévy Processes (spLp) and continuous state branching processes (CSBP)
- there is a forth one T_4 introduced recently by (CPU) between couples of independent Lévy processes (X,Y) and continuous state branching processes with immigration (CBI). Here X is a spLp and Y is a subordinator.

First transformation T_1

Definition

1.- A process X starting at $x=0$ is self-similar if for all $a \geq 0$ there exists $b(a) \in \mathbb{R}$ such that

$$(X_{at}, t \geq 0) \stackrel{\mathcal{L}}{=} (b(a)X_t, t \geq 0)................(\textit{scaling property})$$

2.- A process $(Y_s, s \in \mathbb{R})$ is stationary if $(Y_s, s \in \mathbb{R}) \hat{=} (Y_{s+t}, s \in \mathbb{R})$ for all $t \geq 0$.

Some examples of self-similar processes are: brownian motion, stable Lévy processess, fractional brownian motion....

Theorem

If $(Y_s, s \in \mathbb{R})$ *is a stationary process then for any* $\gamma > 0$ *the process*

$$X_t := t^\gamma Y(logt), \quad t \geq 0, X_0 = 0$$

is self-similar of index γ *and reciprocally if* X *is self-similar of index* $\gamma > 0$*, then*

$$(Y_s, s \in \mathbb{R}) := (e^{-\gamma s} X_{e^s}, s \in \mathbb{R})$$

is stationary.

Maejima, Sato, Samorodniski, Takku, Flandrin, Borgnat, Grey,Rezakhah and the iranian school, have worked on this transformation and it has proven very useful for the study of fractional brownian motion, Ornstein-Uhlenbeck processes, etc. Many more results are obtained when additional conditions are imposed on the self-similar process X: independent increments (sspii) or stationary increments (sspsi).

Self similar process appear as limits of other rescaled Markov processes processes in many situations:

- Brownian Motion and limit of rescaled random walks,
- record process: for $(Y_i)_{i\in\mathbb{N}}$ independent rv with values in $\mathbb{R}^+$ such that $\bar{F}(x) = x^\alpha l(x)$ (l a slowly varying function), Lamperti showed that for $M_n = \{\max Y_i : 0 \leq i \leq n\}$, by rescaling it properly we obtain in the limit a self similar process.
- Galton Watson Processes can be reescaled in otder to get a self similar Marko process (positive)

2.-A positive self-similar Markov process (PSSMP) $\{(X, \mathbb{P}_x), x > 0\}$ is a Markov process with càdlàg paths and values in $[0, \infty]$ which fulfills the scaling property,

$$\textit{The law of } (bX_{b^{-\alpha}t},\, t \geq 0) \textit{ under } \mathbb{P}_x \textit{ is } \mathbb{P}_{bx}. \tag{1}$$

Equivalent to:
The law of $(X_{b^{\alpha}t},\, t \geq 0)$ under $\mathbb{P}_{bx}$ is the same as the law of $(bX_t,\, t \geq 0)$ under $\mathbb{P}_x$.
This holds true for any $x > 0$ and $b > 0$.
The states 0 and ∞ are absorbing states.

Lamperti: there exists a bijection T_1: (LP) $\rightarrow$ (PSSMP) between the set of LP and the set of PSSMP: i.e. If $(X, \mathbb{P}_x)$ is a PSSMP of index $\alpha > 0$ starting at $x > 0$ and

$$S = \inf\{t > 0 : X_t = 0\}$$

then X can be expressed as

$$X_t = x \exp\left\{\xi_{\tau(tx^{-\alpha})}\right\} \qquad 0 \leq t < S\,, \tag{2}$$

with

$$\tau(t) = \inf\left\{s \geq 0 : \int_0^s \exp\{\alpha\xi_u\}\,\mathrm{d}u \geq t\right\}.$$

If $\tau(t) = +\infty$ we take $X_t = 0$.
The process $\xi = (\xi_t,\ t \geq 0)$ is a Lévy process started at 0.
We can also start with the PSSMP X and obtain the LP.

Lamperti also gave the relationship between the generators (Volkonski) If L^X denotes the infinitesimal generator of X and L^ξ that of ξ.

$$\mathcal{L}^\xi f(x) = af'(x)+\frac{\sigma^2}{2}f''(x)+\int_{\mathbb{R}}\big(f(x+y)-f(x)-f'(x)h(y)\big)\Pi(\mathrm{d}y) \tag{3}$$

$$\begin{aligned}\mathcal{L}^{\mathcal{X}} g(z) = a' z^{1-\beta} g'(z) + \frac{\sigma^2}{2} z^{2-\beta} g''(z) \\ + z^{-\beta}\int_0^\infty \big(g(zu) - g(z) - zg'(z)h(\log u)\big)G(\mathrm{d}u).\end{aligned}$$

where $G(\mathrm{d}u) = \Pi(\mathrm{d}u) \circ \log u$, for $u > 0$.

If $\xi = 2B_t + (\delta - 2)t$ with B a Brownian Motion in $\mathbb{R}$. Then X is self-similar of index 1 it is in fact a squared Bessel process . This can be seen by two methods

- analyzing the generators
- using stochastic caluculus

The case when $\xi = qB_t + ct$ $(q = 2, \quad c = (\delta - 2))$:stochastic calculus method: Apply Itô's formula to the function $f(x) = e^x$ and to the process ξ.

$$\exp \xi_t = 1 + q \int_0^t \exp \xi_s dB_s + (q^2/2 + c) \int_0^t \exp \xi_s ds =$$

$$= 1 + q \int_0^t \exp \frac{1}{2} \xi_s dM_s + \delta A_t$$

Where $A_t = \int_0^t \exp \xi_s ds$, $M_t = \int_0^t \exp \frac{1}{2} \xi_s dB_s$, (which is a continuous local martingale) and $A_t = < M >_t$. It is well known that M is a time-changed Brownian motion $(\beta_{A_t} : t \geq 0)$ independent of the former one.

Next we make a change of variable t by $\tau(u)$ in the integral , and use the identify $M_{\tau(v)} = \beta_v$ to get,

$$X_u = \exp \xi_{\tau(u)} = 1 + q \int_0^{\tau(u)} \exp \frac{1}{2} \xi_s dM_t + \delta u =$$
$$1 + q \int_0^u \exp \frac{1}{2} \xi_{\tau(v)} dM_{\tau(v)} + \delta u = 1 + q \int_0^u \exp \frac{1}{2} \xi_{\tau(v)} d\beta_v + \delta u =$$
$$1 + q \int_0^u (X_v)^{1/2} d\beta_v + \delta u$$

These together with Bessel processes are the only continuous PSSMP.
We already saw other examples related to the stable process and we could also get a SDE for them. ...... (work in progress by Leif Doring ans MatyasBarczy)

In a different direction the problem of the entrance law for X was already asked by Lamperti in 1973. This the has been studied by CB, BY , CCH and BS.
The problem raised by Lamperti was : can we make sense of a PSSMP X started at $0+$? Three equivalent ways to see this are:

- problem of the entrance law from $0+$
- asymptotic behaviour of $\mathbb{P}_x$ as $x \to 0$.
- for subordinators: asymptotic behaviour of X_t as $t \to \infty$(scaling property)

This question is important when one analyzes the possible limit processes arising from a sequence of normalized Markov processes.

1.- Brennan-Durrett: (evolution of a particle system) every particle of size $\mathbf{r}$ waits an exponential time of parameter $\mathbf{r}^{\alpha}$ and then splits into a left particle of size $V\mathbf{r}$ and a right particle of size $(1-V)\mathbf{r}$. (V is a r.v. independent of the past behavoiur of the system and with values in $(0,1)$. We focus on the left particle and we write Y_t for its lenght. It is easy to see that it gives rise to a PSSMP $(\frac{1}{Y_t} : t > 0)$.

2.- Problems related to self similar fragmentation.

Theorem

[?] Let ξ be a subordinator with Laplace exponent Φ such that $m = \mathbb{E}(\xi_1) < \infty$. Let X be the PSSMP Lamperti transformation. of ξ. Then for any $t > 0$, $\mathbb{P}_x(X_t \in \cdot)$ converges as $x \to 0+$ to the entrance law at $0+$, denoted $\mathbb{P}_0(X_t \in \cdot)$.

This result was then generalized to the case of any LP ξ with $m = \mathbb{E}(\xi_1) < \infty$ by Bertoin and Yor.
In both cases the fundamental identity is obtained

$$\mathbb{E}_0(X_t^\alpha) = \frac{1}{\alpha t m}, \;\; \mathbb{E}_0(f(X_t^\alpha)) = \frac{1}{\alpha m} E(I^{-1} f(1/I))$$

with $I = \int_0^\infty \exp(-\alpha \xi_s) ds$.
The convergence is at most in the sense of finite dimensional distributions.

The theorem can also be stated in the folowing way : let $\rho = \frac{1}{\alpha}$

Corollary

A ρ-increasing self-similar Markov process X has the following property

$$\frac{X(t)}{t^\rho}$$

converges weakly to a non degenerate rv if the associated LP ξ via the Lamperti transformation has finite mean. It converges to a degenerate rv when ξ has infinite mean.

So it is natural to ask the question of whether or not we can find some result in this case (CR):

Theorem

$$\frac{\log \frac{X(t)}{t^{\rho}}}{\log(t)}$$

converges to a non degenerate rv if and only if the Laplace exponent of the subordinator ξ varies regularly at 0.

Afterwards CCh studied the convergence in the sense of the Skorohod topology and they obtained that convergence of

$$\mathbb{P}_x \to \mathbb{P}_0$$

in the Skorohod topology is equivalent to the convergence of the overshoot process associated to ξ and a certain condition. (2004). They also constructed explicity the limit process $(X_t^{(0)} : t \geq 0)$. which is seen to be a PSSPM
CKPR, were able to supress the additional condition (2009) using a completely different aproach.

Theorem

Let $(X, \mathbb{P}_x)$, $x > 0$ a PSSMP such that the correspondinfg Lp ξ satisfies $\limsup_t \xi_t = +\infty$. The family of measures $\mathbb{P}_x$ converges weakly in the skorohod topology as $x \to 0$ to a non degenerate probablity measure if and only if the overshoot process $(\xi_{T_z} - z, \quad z \geq 0)$ converges weakly to a proper distribution as $z \to +\infty$ and with the additional condition $E(log^+ \int_0^{T_1} \exp \xi_s ds) < \infty$.

Under these conditions the limit law is this of ths process X^0 which will be a positive Markov process with the scaling condition.

The last result in this direction has been obtained in 2010 by Bertoin and Savov [**?**]: first of all they construct a LP on the entire line $(\xi_t : t \in \mathbb{R})$ and then they find a Lamperti Transformation of ths process, which is a PSSMP started from the entrance boundary $0+$. The time change $\sigma(t)$ in this case is defined as

$$\int_{-\infty}^{\sigma(t)} \exp \xi_s ds = t$$

Now it is natural to find a Lamperti Tranformation that permits to construct the version os the PSSMP staring from $0+$. Let $\xi_x = x + \xi$ the version of the LP starting at x and

$$\sigma^x = \inf\{t : \xi^x_t \geq 0\}$$

the first entrance time of the process to $[0, \infty)$. This can be considered as new or extendend Lamperti Transformation.

Third Lamperti transformation : between spectrallly postive Lévy processes (SPLP) and continiuous state branching processesCSBP. We first recall what a CSBP
they are the continuous time and space version of Galton-Watson branching processes and in fact they arrise as rescaled limits of them.
They are Markov process (in fact Feller processes)with space state $[0, \infty]$ and verifying the branching property: the
sum of two independent copies of the process starting at x and at y respectively has the law of the process started at $x + y$. They have a branching mechanism : let Z be a CSBP then

$$\mathbb{E}_z(\exp -\lambda Z_t) = e^{-zu(t,\lambda)}$$

A SPLP ξ, is a Lp without negative jumps and has a Laplace exponent Φ: Let $X = x + \xi$ be the LP started at $x > 0$

$$\mathbb{E}(\exp -\lambda X_t) = \mathbb{E}_x(\exp -\lambda \xi_t) = e^{-\lambda x + t\phi(\lambda)}$$

The correspondence was announced by Lamperti and a version of it was proved by Silverstein, by showing that the branching mechanism of Z corresponds to the Laplace exponent of the SPLP. The version which is best known (but had not a complete proof) is a pathwise method by using the Lamperti time change: The time change is

$$I_t = \int_0^t \frac{1}{x + \xi_{s \wedge \tau}} ds$$

with τ the time when X reaches the negative real line.

Theorem

Given a spectrally positive Lévy process ξ starting at $x > 0$ and killed when entering $(-\infty, 0)$, there exists a CSBP Z with branching mechanism Φ and satisfying

$$Z_t = x + \xi_{C_t}$$

where C_t is the right-continuous inverse of

$$I_t = \int_0^t \frac{1}{x + \xi_{s \wedge \tau}} ds$$

and reciprocally.

This is studied in a survey paper by CLU and furthermore they introduced a generalization of the Skorohod topology which makes this transfomation continuous. On the other hand another proof was given in this same paper using methods of martingales and Stochastic caluculus. This method has been used recently by Leif Doring and Matyas Barczy to study the in general the problems at 0 (work in progress). See also the work by Pardo and Kyprianou in the stable case.

Theorem

(Lamperti representation of CSBP). The Lamperti transformation is a bijection between continuous-state branching processes and Lévy processes with no negative jumps stopped whenever reaching zero. Specically, for any CSBP Z , $T_3(Z)$ is a Lévy process with no negative jumps stopped whenever reaching zero; for any Lévy process with no negative jumps X stopped whenever reaching zero ,$T_3^{-1}(X)$ is a continuous-state branching process.

1- $\xi = qB_t + ct$ has as PSSMP the solution to the SDE
$dZ_t = (Z_t)^{\frac{1}{2}} dB_t + \delta t, \quad (T_2)$
2.-$(x + \xi)$ a SPLP with Lévy triplet (a, σ^2, π) has as CSBP the solution of the SDE

$$Z_t = z + a \int_0^t Z_s ds + \sigma \int_0^t (Z_s)^{\frac{1}{2}} dB_s + \tag{4}$$

$$\int_0^t \int_0^{Z_s -} \int_0^1 u(N - N')(ds, dr, du) + \tag{5}$$

$$\int_0^t \int_0^{Z_s -} \int_0^1 uN(ds, dr, du) \tag{6}$$

where $N')(ds, dr, du) = ds \times dr \times \pi(du)$ is the intensity of the Poisson random measure N independent of B

We will introduce a fourth one, which is new but very much in the spirit of Lamperti , between a couple of LP (X, Y), X beeing a SPLP and Y a subordinator and the class of continuous state branching processes with immigration.
This is joint work with G. Uribe and J.L Pérez Garmendia (work in progress):
The time change is $c(t) = \int_0^t Z_s ds$ and the proposed transformation is a type of Peano equation for cadlag functions:

$$Z_t = X(\int_0^t Z_s ds) + Y_t = X_{c(t)} + Y_t$$

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