Necessary Regular Variation

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Abstract

Regular variation is a convenient description for asymptotic behaviour of functions, allowing a connection to be made between input and output in Abelian or Tauberian contexts. However in some areas regular variation is more than convenient, it is essential, characterising all possible asymptotics for the problem. Examples from probability, complex analysis and number theory will be presented.

1. Regular variation I

Definition 1.1.

A function $f : (a, \infty) \to (0, \infty)$ (where $a \geq 0$) is called *regularly varying* of *index* $\alpha \in \mathbb{R}$, notation $f \in R_\alpha$, if it is measurable and

$$\text{for all } \lambda > 0, \quad \lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha.$$

The *slowly varying* functions are the regularly varying functions of index 0, forming the class R_0.

- Examples of slowly varying functions are all eventually positive rational functions of $\ln = \log_e$ and its iterates.
- ℓ denotes a generic slowly varying function.

Proposition 1.2.

$f \in R_\alpha$ if and only if $\ell(x) := x^{-\alpha} f(x) \in R_0$.

2. Abelian theorems I

A typical *Abelian theorem* gives conditions under which

$$f(x) \sim cx^{\rho}\ell(x) \quad \text{as } x \to \infty \tag{I}$$

implies

$$k \overset{M}{*} f(x) \sim c\check{k}(\rho)x^{\rho}\ell(x) \quad \text{as } x \to \infty. \tag{II}$$

Here the *Mellin transform* of $k : (0, \infty) \to \mathbb{R}$ is given for $z \in \mathbb{C}$, where it exists, by

$$\check{k}(z) := \int_0^{\infty} t^{-z} k(t) \, \frac{dt}{t},$$

and the *Mellin convolution* of two such functions k and f is given, for $x \in \mathbb{R}$, by

$$k \overset{M}{*} f(x) := \int_0^{\infty} k\left(\frac{x}{t}\right) f(t) \, \frac{dt}{t} = \int_0^{\infty} f\left(\frac{x}{t}\right) k(t) \, \frac{dt}{t}.$$

2. Abelian theorems I

Theorem 2.1 ([Arandelović, 1976]).
Let $\check{k}(z)$ exist in the strip $\sigma \le \Re z \le \tau$, where $\sigma < \rho < \tau$, and let $f : (0, \infty) \to \mathbb{R}$ be measurable, with $f(x)/x^\sigma$ bounded on every interval $(0, a]$ for $a > 0$. Then (I) *implies* (II).

Proof.
[Bingham, Goldie & Teugels, 1989, pp. 201–2]. □

In (I) and (II) the constant c can be any real number. All cases $c > 0$ are equivalent, as are all cases $c < 0$. When $c = 0$ the result says that $f(x) = o(x^\rho \ell(x))$ implies $k \overset{M}{*} f(x) = o(\check{k}(\rho) x^\rho \ell(x))$, both as $x \to \infty$.

2. Abelian theorems: Cesàro means I

For $\alpha > 0$ the Cesàro mean of order α of f is given by

$$C_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)x^\alpha} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \quad \text{for } x > 0.$$

Set

$$k(x) := \frac{\mathbf{1}_{[1,\infty)}(x)}{x\Gamma(\alpha)} \left(1 - \frac{1}{x}\right)^{\alpha-1};$$

then $k \overset{M}{*} f = C_\alpha(f)$. This k has Mellin transform

$$\check{k}(z) = \frac{\Gamma(z+1)}{\Gamma(z+\alpha+1)} \quad \text{for } \Re z > -1.$$

Theorem 2.1 thus gives that for all $c \in \mathbb{R}$ and $\rho > -1$, $f(x) \sim cx^\rho \ell(x)$ implies

$$C_\alpha(f)(x) \sim c\frac{\Gamma(\rho+1)}{\Gamma(\rho+\alpha+1)} x^\rho \ell(x) \quad \text{as } x \to \infty.$$

2. Abelian theorems: Cesàro means I

The case $\alpha = 1$ is the familiar 'Cesàro average':

$$C_1(f)(x) = x^{-1} \int_0^x f(t)\, dt.$$

For this case the result is that, again for all $c \in \mathbb{R}$ and $\rho > -1$, $f(x) \sim cx^\rho \ell(x)$ implies

$$C_1(f)(x) \sim \frac{cx^\rho \ell(x)}{(\rho + 1)} \quad \text{as } x \to \infty.$$

2. Abelian theorems: Laplace transform I

Define the Laplace transform $\hat{f}$ by

$$\hat{f}(s) := s \int_0^\infty e^{-st} f(t)\, dt,$$

i.e. with an extra factor s. If $f \in \mathrm{BV}_{\mathrm{loc}}[0,\infty)$ and $f(0-) = 0$ then

$$\hat{f}(s) = \int_{[0,\infty)} e^{-st}\, df(t),$$

so we have defined the Laplace-Stieltjes transform of f. The integral then converges in $\Re s > \sigma$, where possibly $\sigma = \infty$. Set

$$k(x) := x^{-1} e^{-1/x},$$

then $k \overset{M}{*} f(x) = \hat{f}(1/x)$. And

$$\check{k}(z) = \Gamma(1+z) \quad \text{for } \Re z > -1.$$

The Theorem thus gives that for all $c \in \mathbb{R}$ and $\rho > -1$, $f(x) \sim c x^\rho \ell(x)$ implies

$$\hat{f}(s) \sim \frac{c\Gamma(1+\rho)}{s^\rho} \ell\left(\frac{1}{s}\right) \quad \text{as } s \downarrow 0.$$

2. Abelian theorems: power series I

Given coefficients $(a_n)_{n=0}^{\infty}$, let

$$f(x) := \sum_{n=0}^{\lfloor x \rfloor} a_n$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Set $u := e^{-1/x}$ in the latter example; then

$$\hat{f}\left(\frac{1}{x}\right) = \sum_{n=0}^{\infty} a_n u^n.$$

The Theorem thus says that if

$$\sum_{k=0}^{n} a_k \sim c n^{\rho} \ell(n) \quad \text{as } n \to \infty,$$

where $c \in \mathbb{R}$ and $\rho > -1$, then

$$\sum_{n=0}^{\infty} a_n u^n \sim \frac{c\Gamma(1+\rho)}{(-\ln u)^{\rho}} \ell\left(\frac{1}{-\ln u}\right) \quad \text{as } u \uparrow 1.$$

2. Abelian theorems: power series II

Because $-\ln u \sim 1-u$ as $u \uparrow 1$ we may replace $-\ln u$ by $1-u$ in the right-hand side; replacing the argument of ℓ by an asymptotic equivalent involves the Uniform Convergence Theorem for slowly varying functions [Bingham, Goldie & Teugels, 1989, Theorem 1.2.1]. We thus gain the neater conclusion that

$$\sum_{n=0}^{\infty} a_n u^n \sim c\Gamma(1+\rho)\frac{\ell(1/(1-u))}{(1-u)^\rho} \quad \text{as } u \uparrow 1.$$

3. Tauberian theorems I

We want (II) $\Longrightarrow$ (I).

Exercise 3.1.
$(k \overset{M}{*} f)^{\vee}(z) = \check{k}(z)\check{f}(z)$ for $\Re z = \rho$.

So to get information about f from $k \overset{M}{*} f$, need

$$\check{k}(z) \neq 0 \quad \text{for } \Re z = \rho, \tag{W}$$

that is, k is a *Wiener kernel.*

We also need a condition on f. To see this, consider for example the Cesàro mean: if $f(x) = (-1)^{\lfloor x \rfloor}$ then $x^{-1} \int_0^x f \to 0$ as $x \to \infty$, but $f(x) \not\to 0$.

So impose one of

$$\lim_{\lambda \downarrow 1} \liminf_{x \to \infty} \inf_{y \in [x, \lambda x]} \frac{y^{-\rho} f(y) - x^{-\rho} f(x)}{\ell(x)} \geq 0 \qquad (\text{so} = 0), \tag{SD}$$

$$\lim_{\lambda \downarrow 1} \limsup_{x \to \infty} \sup_{y \in [x, \lambda x]} \frac{|y^{-\rho} f(y) - x^{-\rho} f(x)|}{\ell(x)} = 0. \tag{SO}$$

These are extended versions of **slow decrease** (SD) and **slow oscillation** (SO).

Theorem 3.2 ([Bingham & Teugels, 1979]).
Assume the conditions of Theorem 2.1, plus (W), *plus*

- **either** (SO)
- **or** $k \geq 0$ *and* (SD).

Then (II) $\Longrightarrow$ (I).

The case $\ell \equiv 1$ is:

Theorem 3.3 (Wiener-Pitt Theorem).
Assume (W). *If f is bounded and measurable, and of* **slow decrease***:*

$$\lim_{\lambda \downarrow 1} \liminf_{x \to \infty} \inf_{t \in [1,\lambda]} \big(f(tx) - f(x)\big) \geq 0 \quad (hence = 0),$$

then

$$k \overset{M}{*} f(x) \to c\check{k}(0) \quad \textit{implies} \quad f(x) \to c.$$

Example 3.4 (Cesàro means).

An Abel-Tauber theorem: for $\rho > -1$, $f(x) \sim cx^\rho \ell(x)$ as $x \to \infty$ if and only if

$$C_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)x^\alpha} \int_0^x (x-t)^{\alpha-1} f(t)\,dt \sim c\frac{\Gamma(\rho+1)}{\Gamma(\rho+\alpha+1)} x^\rho \ell(x).$$

Example 3.5 (Laplace transforms).

An Abel-Tauber theorem: for $\rho > -1$, $f(x) \sim cx^\rho \ell(x)$ as $x \to \infty$ if and only if

$$\hat{f}(s) := s\int_0^\infty e^{-st} f(t)\,dt \sim \frac{c\Gamma(1+\rho)}{s^\rho} \ell\left(\frac{1}{s}\right) \quad \text{as } s \downarrow 0.$$

3. Tauberian theorems: entire functions I

Example 3.6 (Entire functions).

Let f be entire, with maximal function

$$M(r) := \sup_{|z|\leq r} |f(z)| = \sup_{|z|=r} |f(z)|.$$

Definition 3.7.

The **order** of f is

$$\rho := \limsup_{r\to\infty} \frac{\ln \ln M(r)}{\ln r}.$$

Theorem 3.8 (Proximate Order Theorem [Valiron, 1913]).

If f is entire with order $\rho < \infty$ then there exists $\ell \in R_0$ with

$$\limsup_{r\to\infty} \frac{\ln M(r)}{r^\rho \ell(r)} = 1.$$

3. Tauberian theorems: entire functions I

Definition 3.9.

f has **completely regular growth** if

$$\lim_{r\to\infty}{}^{*} \frac{\ln|f(re^{i\theta})|}{r^{\rho}\ell(r)} = h(\theta) \text{ for all } \theta,$$

where $\lim^*$ means limit as $r \to \infty$ avoiding an exceptional set of density 0.

The zeros of f have **angular density** if

$$\frac{\sum_n \mathbf{1}\{|z_n| \le r,\ \theta \le \arg z_n \le \theta'\}}{r^{\rho}\ell(r)} \to D(\theta, \theta') \quad \text{as } r \to \infty.$$

Levin-Pfluger theory connects these two notions.
The simplest case is when $0 < \rho < 1$. Then

$$f(z) = cz^m \prod_1^n \left(1 - \frac{z}{z_n}\right)$$

where $c \neq 0$, $0 < |z_1| \le |z_2| \le \cdots$.

3. Tauberian theorems: entire functions II

Without loss of generality, take $m = 0$, $c = 1$. Consider the case when the zeros z_1, z_2, ... are *negative reals*. Then

$$\ln f(z) = \int_0^\infty \frac{z/t}{1 + z/t} \, n(t) \, \frac{dt}{t} \quad \text{for } \arg z \neq \pi,$$

where $n(t) := \sum_0^\infty \mathbf{1}\{|z_n| \leq t\}$ is the *zero-counting function*. Then

$$\ln f(re^{i\theta}) = e^{i\theta} k_\theta \overset{M}{*} n(r),$$

where $k_\theta(x) = x/(1 + xe^{i\theta})$, so that

$$\check{k}_\theta(s) = \frac{\pi e^{i\theta(s-1)}}{\sin \pi s} \quad \text{for } 0 < \Re s < 1 \text{ and } \theta \neq \pi.$$

Theorems 2.1 and 3.2 thus give the Levin-Pfluger result that for each $\theta \in (-\pi, \pi)$, $n(r) \sim cr^\rho \ell(r)$ as $r \to \infty$ if and only if

$$\ln f(re^{i\theta}) \sim \frac{c\pi r^\rho e^{i\theta\rho} \ell(r)}{\sin \pi\rho}.$$

3. Tauberian theorems: Lambert kernel I

Example 3.10 (Lambert kernel).

Here

$$k(t) = t\,\frac{d}{dt}\,\frac{1}{t(e^{1/t}-1)}.$$

This has $\check{k}(z) = z\Gamma(1+z)\zeta(1+z)$, non-zero on $\Re z = 0$. Its use is to get a proof of the Prime Number Theorem, as follows.

Definition 3.11.

von Mangoldt's function is

$$\Lambda(x) := \begin{cases} \ln p & \text{if } n = p^k \text{ for some } k = 1,\, 2,\, \ldots, \\ 0 & \text{if not.} \end{cases}$$

One can prove (see for example [Widder, 1941]) that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)-1}{x(e^{n/x}-1)} \to -2\gamma \quad \text{as } x \to \infty.$$

3. Tauberian theorems: Lambert kernel II

The left-hand side is $k \overset{M}{*} f(x)$ where

$$f(x) := \sum_{n=1}^{\lfloor x \rfloor} \frac{\Lambda(n) - 1}{n}.$$

Theorem 3.2 then gives that $f(x) \to -2\gamma$ as $x \to \infty$. This is equivalent (see for example [Hardy & Wright, 1979]) to

Theorem 3.12 (Prime Number Theorem).

$$\sum_{p \le x} 1 \sim \frac{x}{\ln x} \quad \textit{as } x \to \infty.$$

4. Question pause I

Let us ask the following questions:

- Why regular variation?
- Are the conditions right?

We have partly answered the first of these above by giving instances where regular variation plays an important role, necessary for full understanding. We complete our answer in the next Section by giving results from probability theory and analysis where regular variation plays an intrinsic role: it can't be avoided.

We answer the second question in Section 6 by discussing Converse Abelian Theorems.

5. Intrinsic roles: central attraction I

Definition 5.1.

A probability law G is **stable** if there exists a law F such that with X_1, X_2, ... independent $\sim F$, and $S_n := X_1 + \cdots + S_n$, there exist $a_n > 0$ and b_n with

$$\frac{S_n}{a_n} - b_n \xrightarrow{L} G,$$

where $\xrightarrow{L}$ denotes convergence in law. Then we say that F *is attracted to* G.

Theorem 5.2 (Domain of Attraction Theorem).

F is attracted to Gaussian laws if and only if the truncated variance $V(x) := \int_{-x}^{x} t^2 \, dF(t)$ is slowly varying.

F is attracted to a non-Gaussian law G if and only if $1 - F(x) + F(-x) \in R_{-\alpha}$ for some $0 < \alpha < 2$, and there exists

$$\lim_{x \to \infty} \frac{1 - F(x)}{1 - F(x) + F(-x)}.$$

5. Intrinsic roles: extremal attraction I

Definition 5.3.

A probability law G is **extreme-stable** (**extremal**) if there exists a law F such that with X_1, X_2, ... independent $\sim F$, and $M_n = \max(X_1, \dots, X_n)$, there exist $a_n > 0$ and b_n with

$$\frac{M_n}{a_n} - b_n \xrightarrow{L} G.$$

Then we say that $F \in D(G)$.

Theorem 5.4 (Fisher-Tippett-Gnedenko Theorem).

For some $a > 0$, b, $G(ax+b)$ is one of

$$\Phi_\alpha(x) := \begin{cases} 0 & (x < 0), \\ \exp(-x^{-\alpha}) & (x \geq 0), \end{cases} \qquad \textit{where } \alpha > 0;$$

$$\Psi_\alpha(x) := \begin{cases} \exp(-(1-x)^{\alpha}) & (x < 0), \\ 1 & (x \geq 0), \end{cases} \qquad \textit{where } \alpha > 0;$$

$$\Lambda(x) := \exp(-e^{-x}).$$

5. Intrinsic roles: extremal attraction II

Theorem 5.5 (Extremal Attraction Theorem [Gnedenko, 1943, de Haan, 1970]).

$f \in D(\Phi_\alpha)$ if and only if $1 - F \in R_{-\alpha}$.
$f \in D(\Psi_\alpha)$ if and only if $F(x_+) = 1$ and $1 - F(x_+ - x^{-1}) \in R_{-\alpha}$.
$F \in D(\Lambda)$ if and only if $H(x) := -\ln(1 - F(x))$ has inverse $H^{\leftarrow}$ with

$$\lim_{x\to\infty} \frac{H^{\leftarrow}(x+u) - H^{\leftarrow}(x)}{\ell(e^x)} = u \quad \text{for all } u > 0,$$

for some slowly varying ℓ.

5. Intrinsic roles: Mercerian theorems I

If $c \neq 0$, (I) and (II) imply

$$\frac{k \overset{M}{*} f(x)}{f(x)} \to a \quad \text{as } x \to \infty, \tag{III}$$

where $a = \check{k}(\rho)$. Here is a converse:

Theorem 5.6 (Drasin-Shea-Jordan Theorem [Drasin & Shea, 1976, Jordan, 1974]).

Let k be a real kernel and let (a, b) be the maximal open interval such that $\check{k}(z)$ converges absolutely in $a < \Re z < b$. Assume that $\check{k}'(\rho)$ and $\check{k}''(\rho)$ are not both 0, that k is monotone on $[\rho, b)$ and zero on $(0, 1)$, and that

$$\check{k}(z) \neq \check{k}(\rho) \text{ for } \Re z = \rho \text{ and } z \neq \rho.$$

Let $f \geq 0$ be locally bounded on $[0, \infty)$, have finite order $\rho \in (a, b)$, and have **bounded decrease***:*

$$\liminf_{x\to\infty} \inf_{\mu\in[1,\lambda]} \frac{f(\mu x)}{f(x)} > 0$$

for some (equivalently all) $\lambda > 1$. Then (III) implies $a = \check{k}(\rho)$ and $f \in R_\rho$.

6. Converse Abelian theorems I

The Wiener-Pitt Theorem needs the Wiener condition (W) on k, and for f to be locally bounded and of slow decrease. The extra condition for Theorem 3.2 is

$$\check{k}(z) \text{ exists for } \sigma \le \Re z \le \tau, \text{ for some } \sigma < \rho < \tau.$$

This cannot be omitted:

Theorem 6.1 (Converse Abelian Theorem [Arandelović, 1976]).

Let $R^o_\rho := \{f \in R_\rho : f \text{ locally bounded on } (0,\infty), \, \mathrm{O}(x^\rho) \text{ as } x \downarrow 0\}$. *The following are equivalent:*

$$k \overset{M}{*} f(x) = \mathrm{O}\big(f(x)\big) \text{ as } x \to \infty, \text{ for all } f \in R^o_\rho;$$

$$\check{k}(z) \text{ exists for } \rho - \delta \le \Re z \le \rho + \delta, \text{ for some } \delta > 0.$$

The proof needs:

Proposition 6.2 ([Vuilleumier, 1963]).

If f is such that $f(x)\ell(x) = \mathrm{O}(1)$ as $x \to \infty$, for every non-decreasing slowly varying ℓ, then $x^\alpha f(x) = \mathrm{O}(1)$ as $x \to \infty$, for some $\alpha > 0$.

6. Converse Abelian theorems II

Proof.

Let us show that if $\limsup_{x\to\infty} x^{\alpha}|f(x)| = \infty$ for each $\alpha > 0$, then also $\limsup_{x\to\infty} \ell(x)|f(x)| = \infty$ for some non-decreasing $\ell \in R_0$.

Set $a_0 := 1$.

Because $\limsup x^{1/k}|f(x)| = \infty$ for each $k = 1, 2, \ldots$, we may successively find $a_1, a_2, \ldots$ such that $a_k \geq a_{k-1} + 1$ and $a_k^{1/k}|f(a_k)| \geq k$ for $k = 1, 2, \ldots$.

Define $\varepsilon(a_k) := 1/k$, and complete $\varepsilon(x)$ so as to be continuous and piecewise-linear.

Then $\varepsilon(x) \downarrow 0$ as $x \to \infty$, while $\limsup_{x\to\infty} x^{\varepsilon(x)}|f(x)| = \infty$.

Set $\ell(x) := \exp \int_1^x \varepsilon(y) y^{-1}\, dy$, then ℓ is slowly varying, and

$$\ell(x)|f(x)| = |f(x)| \exp \int_1^x \varepsilon(y)\, \frac{dy}{y} \geq x^{\varepsilon(x)}|f(x)|$$

is unbounded as $x \to \infty$. $\square$

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