Non-convergent extremes, coupon collecting and computer-based tests

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Extremes, coupons, tests

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2. Coupon collecting
3. Computer-based tests
4. N_q for specific values of a & q
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1. Extremes I

$Y, \ Y_1, \ Y_2$, i.i.d. $\tilde{F}$.

$$¿\,\exists \ a_n > 0, \ b_n \text{ so that } \frac{\max(Y_1, \dots, Y_n) - b_n}{a_n} \implies \text{non-degenerate limit?}$$

Necessary for weak convergence (convergence in law) that

$$\frac{1 - F(x)}{1 - F(x-)} \to 1 \text{ as } x \to \infty.$$

So if Y discrete, with probabilities geometrically decaying:

$$\frac{P(Y = k)}{P(Y = k+1)} \to c > 1,$$

weak convergence of $\max(Y_1, \dots, Y_n)$, however centred & normed, can't occur [Anderson, 1970].

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2. Coupon collecting I

There are a types of coupon. Each cereal packet has one.
$Y :=$ # packets needed to get at least 1 coupon of each type.

$$Y = X_1 + X_2 + \cdots + X_a,$$
$$X_1, \ X_2 \text{ independent}, \ X_k \sim \text{Geom}_1\left(\frac{a-k+1}{a}\right),$$

where the Geom_1 law has probabilities $p(1-p)^{k-1}$ at $k = 1, 2, \dots$.

2. Coupon collecting I

Law of Y

Let the coupon types be 1, . . . , a. Let

$$A_i := \{\text{type } i \text{ doesn't occur in the first } y \text{ cereal packets bought.}\}$$

So $\{Y > y\} = A_1 \cup A_2 \cup \cdots \cup A_a$, hence

$$\begin{aligned}
&P(Y > y)\\
&= \sum_1^a P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \\
&\qquad \cdots + (-)^{a+1} P(A_1 \cap \cdots \cap A_a)\\
&= \sum_1^a \left(1 - \frac{1}{a}\right)^y - \sum_{i<j} \left(1 - \frac{2}{a}\right)^y + \sum_{i<j<k} \left(1 - \frac{3}{a}\right)^y - \cdots + (-)^{a+1} \left(1 - \frac{a}{a}\right)^y\\
&= \sum_{k=1}^a (-)^{k+1} \binom{a}{k} \left(1 - \frac{k}{a}\right)^y,
\end{aligned}$$

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2. Coupon collecting II

This formula,

$$P(Y > y) = \sum_{k=1}^a (-)^{k+1} \binom{a}{k} \left(1 - \frac{k}{a}\right)^y,$$

is a classical one for the probability that not all cells are occupied when y balls are distributed at random among a cells.

For large y the 1$^{\text{st}}$ term dominates, i.e.

$$P(Y > y) \sim a\left(1 - \frac{1}{a}\right)^y \text{ as } y \to \infty \ (y \in \mathbb{N},\ a \text{ fixed}).$$

3. Computer-based tests I

Each student takes a test of q questions.

For each question there is a bank of a alternatives.

The computer generates a test by selecting, for each of the q questions, one of the a alternatives for that question.

Let $N_q :=$ # tests one needs to generate to see all aq alternatives in the q question banks at least once.

I fix a, for instance $a := 10$, and consider how N_q behaves for various q.

The case $q = 1$, i.e. a 1-question test, is coupon-collecting.

Coupon-collecting asymptotics are for $Y = N_1$ as $a \to \infty$, but I'm interested in N_q as q grows, for fixed a.

The case considered is coupon collecting when q brands of cereal bought simultaneously, each brand having a different set of a coupons to collect. Therefore

$$N_q = \max(Y_1, \ldots, Y_q)$$

where the Y_i are independent with the coupon-collecting distribution.

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4. N_q for specific values of a & q I

$$\begin{aligned} EN_q &= \sum_{n=0}^{\infty} P(N_q > n) \\ &= \sum_{n=0}^{\infty} \Big(1 - \prod_{i=1}^{q} P(Y_i \le n)\Big) \\ &= \sum_{n=0}^{\infty} \Big(1 - \big(1 - P(Y > n)\big)^q\Big). \end{aligned}$$

This formula involves an infinite sum. Although one may derive a formula with no infinite sum,

$$EN_q = -\sum_{m=1}^{q} \binom{q}{m} \sum_{j_1=1}^{a} \cdots \sum_{j_m=1}^{a} \frac{(-1)^{j_1+\cdots+j_m} \binom{a}{j_1} \cdots \binom{a}{j_m}}{1 - \prod_{i=1}^{m}(1 - j_i/a)},$$

it is interesting that the latter is much less suited to explicit calculation, and in what follows we have used the first formula, with its infinite sum.

4. N_q for specific values of a & q II

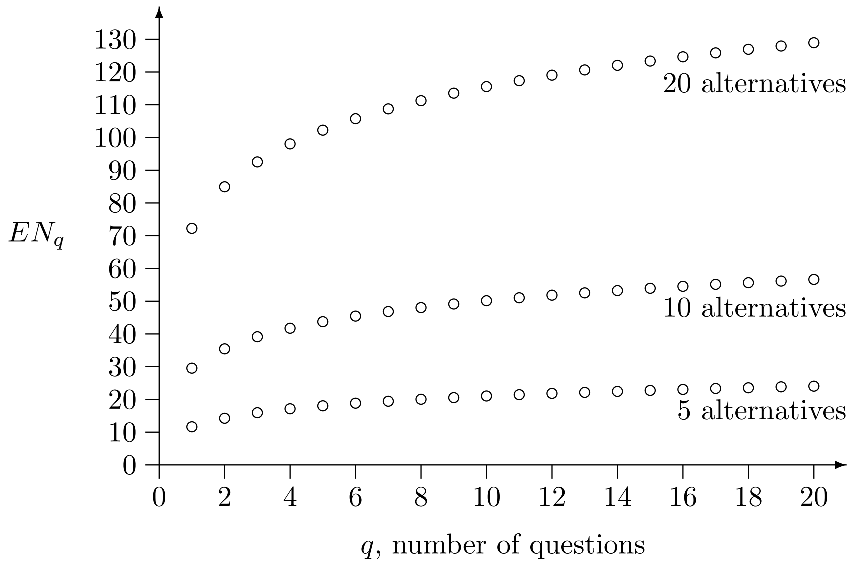

Figure 1: EN_q, the expected number of tests that need to be generated in order for all questions to have appeared at least once, for tests with up to 20 questions and 5, 10, and 20 alternatives for each question.

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4. N_q for specific values of a & q III

Note that in a 20-question test with 5 alternatives for each question, there are $5^{20} = 95\,367\,431\,640\,625$ different possible tests and a total bank of 100 questions; however, on average all questions will have appeared at least once by the time only 24 tests have been generated.

4. N_q for specific values of a & q I

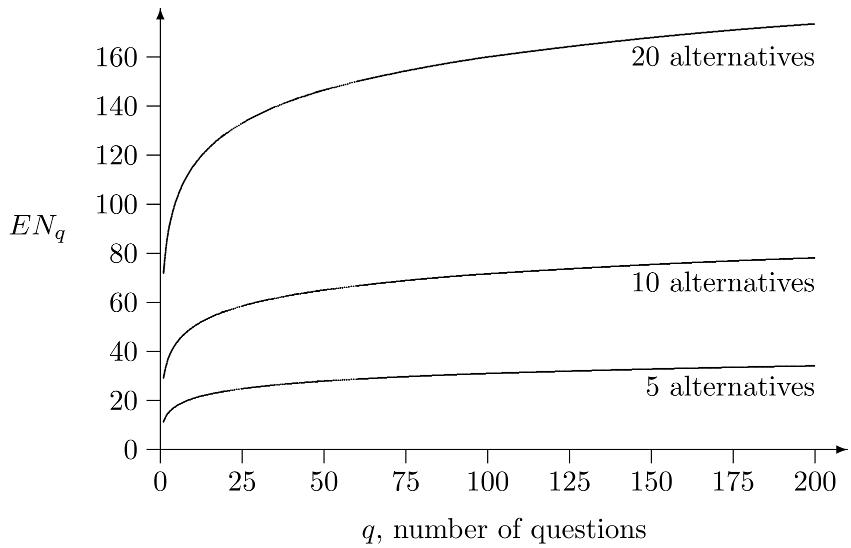

Figure 2: EN_q, the expected number of tests that need to be generated in order for all questions to have appeared at least once, for tests with up to 200 questions and 5, 10, and 20 alternatives for each question.

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5. N_q for q large I

Set $\alpha := \ln \frac{a}{a-1} > 0$, then I showed

$$P(Y > y) \sim ae^{-\alpha y} \text{ as } y \to \infty, \ \ y \in \mathbb{N}.$$

Ignoring the restriction to $\mathbb{N}$,

$$\begin{aligned} P\Big(N_q - \frac{\ln(aq)}{\alpha} \le x\Big) &= \Big(P\Big(Y \le \frac{\ln(aq)}{\alpha} + x\Big)\Big)^q \\ &= \big(1 - ae^{-\ln(aq) - \alpha x}(1 + \mathrm{o}(1))\big)^q \\ &= \Big(1 - \frac{e^{-\alpha x}(1 + \mathrm{o}(1))}{q}\Big)^q \to e^{-e^{-\alpha x}} = \Lambda(\alpha x), \end{aligned}$$

where $\Lambda(x) := e^{-e^{-x}}$ is the Gumbel distribution function.

Theorem 1.
With $b_q := \frac{1}{\alpha}\ln(aq)$,

$$\liminf_{q\to\infty} P(N_q - b_q \le x) = \Lambda\big(\alpha(x-1)\big);$$

$$\limsup_{q\to\infty} P(N_q - b_q \le x) = \Lambda(\alpha x).$$

5. N_q for q large II

Thus $N_q - b_q$ is, asymptotically, in distribution between $\frac{Z}{\alpha}$ and $\frac{Z}{\alpha} + 1$ where Z Gumbel, and the bounds are sharp.

Let $\lfloor x \rfloor$ denote the integer part, $\{x\} := x - \lfloor x \rfloor$ the fractional part, of x.

Theorem 2 (extending [Anderson, 1980, Ferguson, 1993]).

$$P(N_q - b_q = n+1-\{b_q\}) = P\Big(\frac{Z}{\alpha} \le n+1-\{b_q\}\Big) - P\Big(\frac{Z}{\alpha} \le n-\{b_q\}\Big) + o_n(1),$$

where $\sum_{n\in\mathbb{Z}} o_n(1) \to 0$ *as* $q \to \infty$.

Extremes, coupons, tests

6. L_p-boundedness I

Theorem 3.
$N_q - b_q$ is L_p-bounded for all p, i.e. $\sup_{q\in\mathbb{N}} E\big(|N_q - b_q|^p\big) < \infty$ *for all* $p \ge 1$.

Proof.
Fix $n \in \mathbb{N}$; set $R_q := N_q - b_q$. I prove $\sup_q E(R_q^{2n}) < \infty$, which suffices. Now

$$E(R_q^{2n}) = -2n\int_{-\infty}^{0} x^{2n-1} P(R_q \le x)\,dx + 2n\int_0^\infty x^{2n-1} P(R_q > x)\,dx =: A + B.$$

For B, show

$$P(Y > x + b_q) \le \frac{2}{q} e^{\alpha - \alpha x} \quad \forall x \ge 0,\ q \ge q_0;$$

$$\therefore\ P(R_q > x) \le 1 - \Big(1 - \frac{2}{q} e^{\alpha-\alpha x}\Big)^q \le 4e^{\alpha-\alpha x} \quad \forall\ x \ge 0,\ q \ge q_1;$$

$$\therefore\ B \le 8n\int_0^\infty x^{2n-1} e^{\alpha-\alpha x}\,dx < \infty.$$

For A, adapt a split-and-bound technique from [Resnick, 1987]. □

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7. Mean growth I

L_p-boundedness implies asymptotic bounds on moments. Recall

$$\frac{Z}{\alpha} \le N_q - b_q \le \frac{Z}{\alpha} + 1 \quad \text{asymptotically},$$

and $EZ = \gamma \doteq 0{\cdot}5772$.

Theorem 4.

$$\frac{\gamma}{\alpha} \le \lim_{q\to\infty} \sup/\inf (EN_q - b_q) \le \frac{\gamma}{\alpha} + 1.$$

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7. Mean growth I

q	1	10	100	1000
EN_q	29·29	49·90	71·57	93·40
$b_q + \gamma/\alpha$	27·33	49·19	71·04	92·90
excess	1·956855	0·715025	0·527514	0·503224

q	10 000	10^5	10^6	10^7
EN_q	115·25	137·10	158·96	180·81
$b_q + \gamma/\alpha$	114·75	136·60	158·46	180·31
excess	0·500358	0·500039	0·500004	0·500000

Table 1: For $a = 10$, values of EN_q, its approximant $b_q + \gamma/\alpha$, and the excess $EN_q - (b_q + \gamma/\alpha)$.

Conjecture.
As $q \to \infty$, $EN_q - b_q - \frac{\gamma}{\alpha} \to$ limit, maybe $0{\cdot}5$.

8. Variance stability I

Lemma.

$$E\left(\left(1+\frac{Z}{\alpha}\right)^2 \mathbf{1}_{1+\alpha^{-1}Z\le 0} + \left(\frac{Z}{\alpha}\right)^2 \mathbf{1}_{Z>0}\right) \le \limsup_{q\to\infty}/\inf E\big((N_q - b_q)^2\big) \le E\left(\left(\frac{Z}{\alpha}\right)^2 \mathbf{1}_{Z\le 0} + \left(1+\frac{Z}{\alpha}\right)^2 \mathbf{1}_{1+\alpha^{-1}Z>0}\right)$$

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8. Variance stability I

Note $\operatorname{var} Z = \pi^2/6$, so without discreteness we'd get $\operatorname{var} N_q \to \frac{\pi^2}{6\alpha^2}$.

Theorem 5.

$$\limsup_{q\to\infty}\left|\operatorname{var} N_q - \frac{\pi^2}{6\alpha^2}\right| \le \theta(\alpha) + 1 - \frac{1}{e} + \frac{2(\gamma + E_1(1))}{\alpha},$$

where

$$\theta(\alpha) = E\left(\left(1+\frac{Z}{\alpha}\right)^2 \mathbf{1}_{0<1+\alpha^{-1}Z\le 1}\right) \in (0,1),$$

$$E_1(1) = \int_1^\infty \frac{e^{-t}}{t}\,dt \doteq 0{\cdot}2194.$$

a	2	3	4	5	10	20
sd(N_q)	1·873	3·176	4·468	5·755	12·176	25·006
$\pi/(\alpha\sqrt{6})$	1·850	3·163	4·458	5·748	12·173	25·004
Min s.d.	0·641	2·323	3·697	5·024	11·507	24·362
Max s.d.	2·537	3·823	5·107	6·390	12·804	25·630

Table 2: Asymptotic standard deviation of N_q, its approximant, and bounds.

8. Variance stability II

Conjecture.
As $q \to \infty$, $\operatorname{var} N_q \to$ *limit.*

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