Lifts and generalized morphisms

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joint work with
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Overview

Computable ring

Definition ([BLH11, Def. 3.2])

A ring R is called left (resp. right) **computable** if one can specify an algorithm to solve inhomogeneous linear equations $B = XA$ (resp. $B = AX$) over R.

In other words, we want to be able to compute a generating set of syzygies and to effectively decide solvability, i.e., to compute a particular solution.

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Computable ABELian categories

Definition ([BLH11, Def. 2.1])

An ABELian category is called **computable**[a] if the existential quantifiers entering the defining axioms can be turned into constructive ones.

[a] as an ABELian category.

$R - \mathrm{fpmod}$ over a computable ring is computable

Theorem ([BLH11, Thm. 3.4])

The category $R - \mathrm{fpmod}$ of finitely presented left (resp. right) modules over a left (resp. right) computable ring R is ABEL*ian and, as such, computable.*

Below we list the two additional axioms an additive category needs to satisfy to become ABELian.

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$\mathcal{A}$ is an ABELian category

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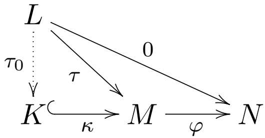
10. For any morphism $\varphi : M \to N$ there exists a **kernel** $\ker\varphi \stackrel{\kappa}{\hookrightarrow} M$, such that
11. for any morphism $\tau : L \to M$ and any monomorphism $\kappa : K \hookrightarrow M$ with $\tau\varphi = 0$ for $\varphi = \operatorname{coker}\kappa$ there exists a **lift** $\tau_0 : L \to K$ of τ along κ.
12. For any morphism $\varphi : M \to N$ there exists a **cokernel** $N \stackrel{\varepsilon}{\twoheadrightarrow} \operatorname{coker}\varphi$, such that
13. for any morphism $\eta : N \to L$ and any epimorphism $\varepsilon : N \twoheadrightarrow C$ with $\varphi\eta = 0$ for $\varphi = \ker\varepsilon$ there exists a **colift** $\eta_0 : C \to L$ of η along ε.

A morphism $\kappa : K \to M$ is called "the" **kernel** of $\varphi : M \to N$ if

(i) $\kappa\varphi = 0$, and

(ii) for all objects L and all morphisms $\tau : L \to M$ with $\tau\varphi = 0$ there exists a *unique* morphism $\tau_0 : L \to K$, such that $\tau = \tau_0\kappa$. τ_0 is called the **lift** of τ along κ.

It follows from the uniqueness of the lift τ_0 that κ is a *monomorphism*.



K is called "the" **kernel object** of φ. This funny diagram just means that

$$\operatorname{im}\tau \leq \operatorname{im}\kappa,$$

in the categorial language.

$\mathtt{X} = \mathtt{SyzygiesGenerators}(\mathtt{A})$

Let $\mathtt{A}$ be an $r_1 \times r_0$-matrix over R.

We call $\mathtt{X} \in R^{r_2 \times r_1}$ a matrix of **generating syzygies (of the rows) of** $\mathtt{A}$ if for all $\mathtt{x} \in R^{1 \times r_1}$ with $\mathtt{xA} = 0$, there exists a $\mathtt{y} \in R^{1 \times r_2}$ such that $\mathtt{yX} = \mathtt{x}$. The rows of $\mathtt{X}$ are thus a generating set of the kernel of the map $R^{1 \times r_1} \xrightarrow{\mathtt{A}} R^{1 \times r_0}$. We write

$$\mathtt{X} = \mathtt{SyzygiesGenerators}(\mathtt{A})$$

and say that $\mathtt{X}$ is the most general solution of the homogeneous linear system $\mathtt{XA} = 0$.

$\mathtt{X} = \mathtt{RelativeSyzygiesGenerators}(\mathtt{A}, \mathtt{L})$

Further let $\mathtt{L}$ be an $r_1' \times r_0$-matrix over R. We call $\mathtt{X} \in R^{r_2 \times r_1}$ a matrix of **relative generating syzygies (of the rows) of** $\mathtt{A}$ **modulo** $\mathtt{L}$ if the rows of $\mathtt{X}$ form a generating set of the kernel of the map $R^{1\times r_1} \xrightarrow{\mathtt{A}} \operatorname{coker} \mathtt{L}$. We write

$$\mathtt{X} = \mathtt{RelativeSyzygiesGenerators}(\mathtt{A}, \mathtt{L})$$

and say that $\mathtt{X}$ is the most general solution of the homogeneous linear system $\mathtt{XA} + \mathtt{YL} = \mathtt{0}$. This last system is of course equivalent[1] to solving the homogeneous linear system

$$\begin{pmatrix} \mathtt{X} & \mathtt{Y} \end{pmatrix} \begin{pmatrix} \mathtt{A} \\ \mathtt{L} \end{pmatrix} = \mathtt{0}.$$

[1] In practice, however, one can often implement efficient algorithms to compute $\mathtt{X}$ without explicitly computing $\mathtt{Y}$.

How to compute $\ker\varphi \overset{\kappa}{\hookrightarrow} M$ of $\varphi : M \to N$?

In the following we take $M := \operatorname{coker} \mathtt{M}$ and $N := \operatorname{coker} \mathtt{N}$.

How to compute $\ker\varphi \overset{\kappa}{\hookrightarrow} M$ of $\varphi : M \to N$?

To compute the kernel $\ker\varphi \overset{\kappa}{\hookrightarrow} M$ of a morphism $\varphi : M \to N$ represented by a matrix $\mathtt{A}$ we do the following:

1. First compute

$$\mathtt{X} = \mathtt{RelativeSyzygiesGenerators}(\mathtt{A}, \mathtt{N}),$$

   the matrix representing κ.
2. Then $\ker\varphi$ is presented by the matrix

$$\mathtt{K} = \mathtt{RelativeSyzygiesGenerators}(\mathtt{X}, \mathtt{M}).$$

$\mathtt{X} = \mathtt{RightDivide}(\mathtt{B}, \mathtt{A})$

Further let $\mathtt{B}$ be an $r_2 \times r_0$-matrix over R.

Deciding the solvability and solving the inhomogeneous linear system $\mathtt{XA} = \mathtt{B}$ is equivalent to the construction of matrices $\mathtt{N}, \mathtt{T}$ such that $\mathtt{N} = \mathtt{TA} + \mathtt{B}$ satisfying the following condition: If the i-th row of $\mathtt{B}$ is a linear combination of the rows of $\mathtt{A}$, then the i-th row of $\mathtt{N}$ is zero[2]. Hence the inhomogeneous linear system $\mathtt{XA} = \mathtt{B}$ is solvable (with $\mathtt{X} = -\mathtt{T}$), if and only if $\mathtt{N} = \mathtt{0}$. We write

$$(\mathtt{N}, \mathtt{T}) = \mathtt{DecideZeroEffectively}(\mathtt{A}, \mathtt{B}) \text{ and } \mathtt{N} = \mathtt{DecideZero}(\mathtt{A}, \mathtt{B}).$$

In case $\mathtt{N} = \mathtt{0}$ we write

$$\mathtt{X} = \mathtt{RightDivide}(\mathtt{B}, \mathtt{A}).$$

[2] So we do not require a “normal form”, but only a mechanism to decide if a row is zero modulo some relations.

submodule membership problem

Rows of the matrices $\mathtt{A}$ and $\mathtt{B}$ can be considered as elements of the free module $R^{1 \times r_0}$.

- Deciding the solvability of the inhomogeneous linear system $\mathtt{XA} = \mathtt{B}$ for a single row matrix $\mathtt{B}$ is thus nothing but the **submodule membership problem** for the submodule generated by the rows of the matrix $\mathtt{A}$.
- Finding a particular solution $\mathtt{X}$ (in case one exists) solves the submodule membership problem **effectively**.

$\mathtt{X} = \mathtt{RightDivide}(\mathtt{B}, \mathtt{A}, \mathtt{L})$

As with relative syzygies we also consider a relative version. In case the inhomogeneous system $\mathtt{XA} = \mathtt{B} \mod \mathtt{L}$ is solvable, we denote a particular solution by

$$\mathtt{X} = \mathtt{RightDivide}(\mathtt{B}, \mathtt{A}, \mathtt{L}).$$

This is equivalent to solving

$$\begin{pmatrix} \mathtt{X} & \mathtt{Y} \end{pmatrix} \begin{pmatrix} \mathtt{A} \\ \mathtt{L} \end{pmatrix} = \mathtt{B}.$$

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Let $\tau : L \to M$ be a morphism represented by a matrix $\mathtt{B}$ and $\kappa : K \hookrightarrow M$ a monomorphism represented by a matrix $\mathtt{A}$ with $\tau\varphi = 0$ for $\varphi = \operatorname{coker} \kappa$. Then the matrix

$$\mathtt{X} = \mathtt{RightDivide}(\mathtt{B}, \mathtt{A}, \mathtt{M})$$

is a representation matrix for $\tau_0 : L \to K$, the lift of τ along κ.

It is an easy exercise[3] to check that $\mathtt{X}$ represents a *morphism*.

[3] Cf. [BR08, 3.1.1, case (2)]).

Applications of lifts

Applications of the lift in ABELian categories

- Compute the morphism part of a functor, e.g.,

$$\mathrm{Ext}^c_R(N,L) \xrightarrow{\mathrm{Ext}^c_R(\varphi,L)} \mathrm{Ext}^c_R(M,L),$$

for $\varphi : M \to N$.
- ...

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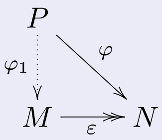
$\mathcal{A}$ has **enough projectives**:

14. For each morphism $\varphi : P \to N$, with P projective, and each morphism $\varepsilon : M \to N$ with $\operatorname{im}\varphi \leq \operatorname{im}\varepsilon$ there exists a **projective lift** $\varphi_1 : P \to M$ of a φ along ε.
15. For each object M there exists a **projective hull** $\nu : P \twoheadrightarrow M$.

Projective object and projective lift

Definition

An object P in a category $\mathcal{A}$ is called **projective**, if for each epimorphism $\varepsilon : M \twoheadrightarrow N$ and each morphism $\varphi : P \to N$ there exists a morphism $\varphi_1 : P \to M$ with $\varphi_1\varepsilon = \varphi$.



We call φ_1 a **projective lift** of φ along ε.

Deciding projectiveness in $R - \mathbf{fpmod}$

We already know several methods to test projectiveness in $R - \mathbf{fpmod}$.

- At least one of them does not make use of syzygies (needs commutativity).
- Another one uses Ext^1_R (needs commutativity).
- The FITTING criterion: $\mathrm{Fitt}\, M = R$ (needs commutativity).
- And two that do not need the commutativity assumption, but the computability of a finite free resolution instead [AB69, CQR05] and [Lam06, QR07].

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Deciding projectiveness in $R-\mathrm{fpmod}$ using a split

Let $\nu : F_0 \twoheadrightarrow M$ be a free presentation of the R-module M. It follows that M is projective if and only if ν admits a section $\sigma : M \hookrightarrow F_0$ (i.e., $\sigma\nu = \mathrm{id}_M$).

Deciding projectiveness in $R-\mathrm{fpmod}$ without syzygies

Finding the section σ for a finitely and freely presented module $M \overset{\nu}{\twoheadleftarrow} F_0 \overset{\mathtt{M}}{\leftarrow} F_1$ leads to solving the two-sided inhomogeneous linear system

$$\mathtt{X} + \mathtt{Y}\mathtt{M} = \mathtt{Id}, \ \mathtt{M}\mathtt{X} = 0,$$

over R, where $\mathtt{X}$ is a square matrix representing σ and $\mathtt{Y}$ another unknown matrix.

This system can be easily brought to a one-sided inhomogeneous linear system if R is commutative[4].

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Again let $\nu : F_0 \twoheadrightarrow M$ be a free presentation of M.

Theorem

An R-module M is projective if and only if $\mathrm{Ext}^1_R(M, K_1(M)) = 0$.

Proof.

$\mathrm{Ext}^1(M, K_1(M)) = 0$ implies that the extension $M \twoheadleftarrow F_0 \hookleftarrow K_1(M)$ splits, i.e., $F_0 \cong M \oplus K_1(M)$ and M is projective as a direct summand of the free module F_0. □

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Theorem ([Eis95, Prop. 20.8])

*A finitely presented R-module M over a commutative ring R is projective of **constant rank** if and only if $\mathrm{Fitt}\, M = R$.*

Example (Caution)

Take $R := k[x]/\langle x^2 - x\rangle \cong k \times k$ and

$$I := \langle x\rangle \cong k \oplus 0 \lhd R.$$

I, as an R-module, is projective but not of constant rank (its (global) rank is 0).

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Deciding freeness – QUILLEN-SUSLIN

Deciding freeness seems to be much harder than deciding projectiveness.

Theorem (QUILLEN-SUSLIN)

If R is a commutative principal ideal domain then $R[x_1, \ldots, x_n]$ is a HERMITE ring.

See [FQ07] for an implementation.

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Deciding freeness – Hairy ball theorem

Example (Hairy ball theorem)

Let $R = \mathbb{R}[x,y,z]/\langle x^2+y^2+z^2-1\rangle$. Then

$$R^{1\times 3}/\langle \begin{pmatrix} x & y & z \end{pmatrix} \rangle$$

is stably free but not free.

Proof.

This follows from the hairy ball theorem in analysis. □

There is no known[5] *algebraic* proof for non-freeness!

[5] July 2011

Software demo

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How to compute the **free** lift $\varphi_1 : P \to M$?

Let F be a **free** R-module presented by an empty matrix, i.e., F is given on a set of *free* generators. Further let $\varphi : F \to N$ and $\varepsilon : M \to N$ be morphisms represented by the matrices `B` and `A`, respectively.

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The image condition $\operatorname{im}\varphi \leq \operatorname{im}\varepsilon$ guarantees the existence of the matrix

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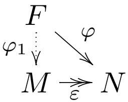

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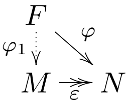

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Applications of free lifts

Applications of the free lift in ABELian categories

- Resolution of morphisms.
- Horseshoe lemma and Cartan-Eilenberg resolution.
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Overview

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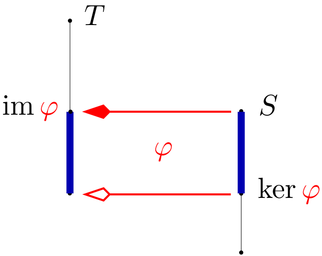

Figure: The homomorphism theorem of a morphism

Subfactors as images?

How to relate **subfactor** objects,e.g., (co)homologies, to their hull objects in a categorical way?

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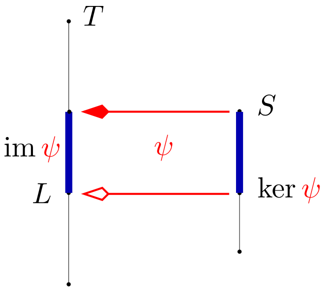


Figure: A homomorphism theorem generalized morphism

Cf. [Bar, Def. 4.1].

HASSE diagram of a generalized morphism

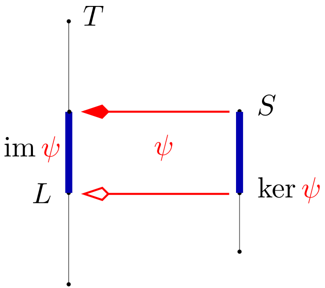


Figure: A homomorphism theorem generalized morphism

Cf. [Bar, Def. 4.1].

Composition of generalized morphisms

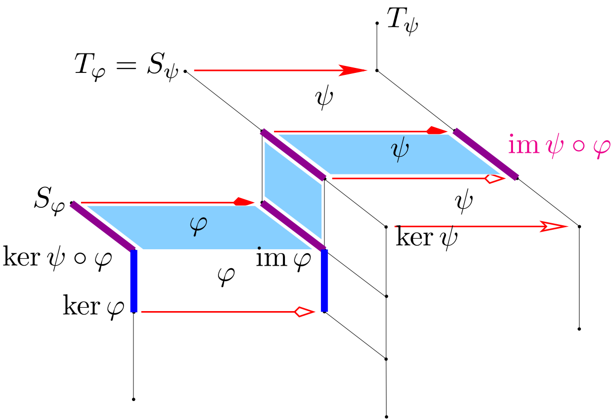


Figure: The composition $\psi\circ\varphi$

The lifting emma

Lemma (The lifting lemma [Bar, Lemma 4.5])

Let $\gamma = (\bar{\gamma}, L_\gamma)$ and $\beta = (\bar{\beta}, L_\beta)$ be two generalized morphisms with the same target N. Suppose that β lifts γ. Then there exists a generalized morphism $\alpha : M' \to N'$ with $\beta \circ \alpha \triangleq \gamma$,

$$\begin{array}{ccc} M' & & \\ {\scriptstyle\alpha}\downarrow & \searrow^{\gamma} & \\ N' & \xrightarrow[\beta]{} & N. \end{array}$$

$$\vdots$$

Lifting of generalized morphisms

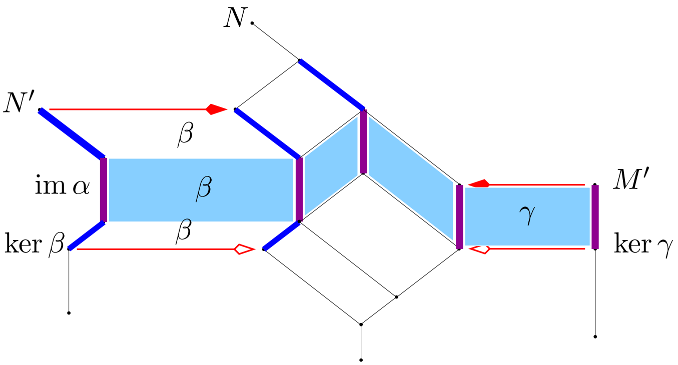
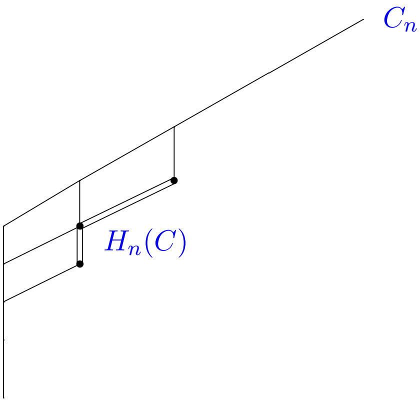


Figure: The lifting condition and the lifting lemma

The spectral filtration of $H(C_n)$

Q: What happens inside the object C_n while flipping the pages?



A: $H_n(C)$ got approximated and the homomorphism theorem can be used to recover the extension [Bar].

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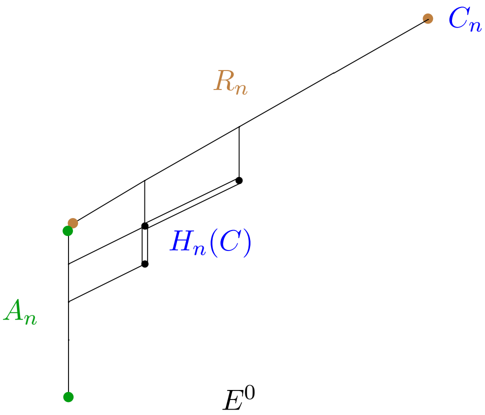

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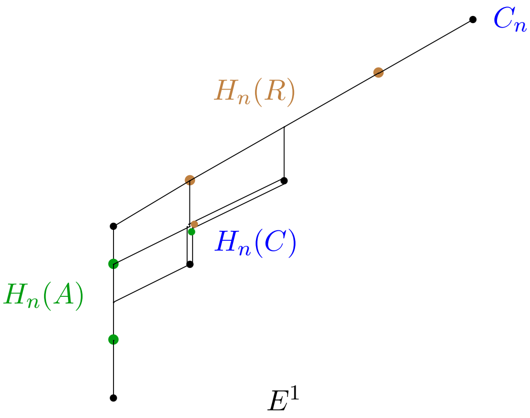

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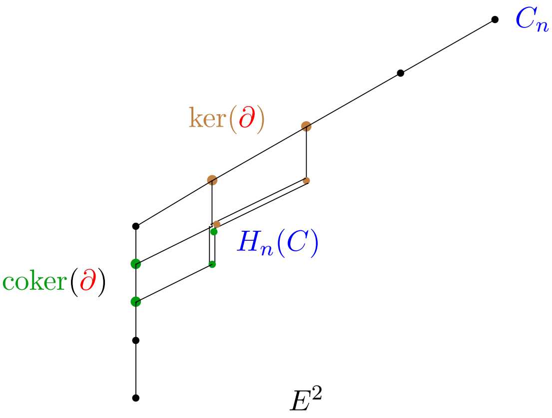

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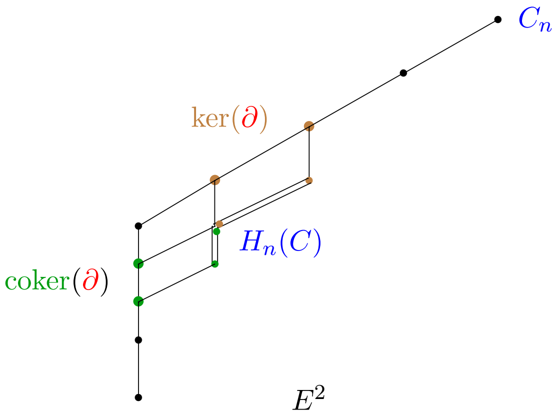

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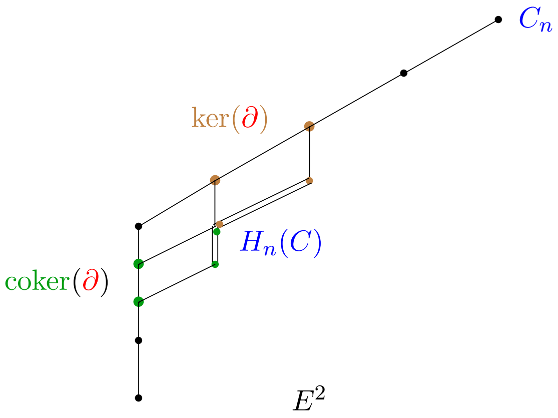

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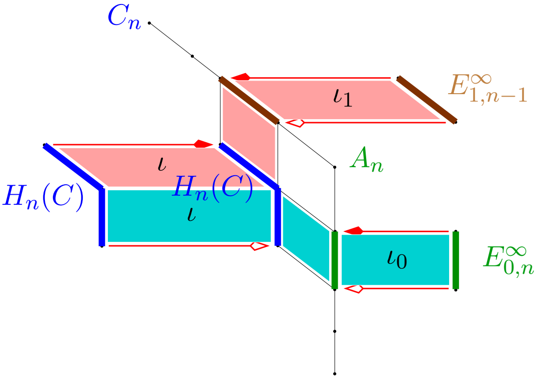

Figure: ι lifts ι_0 and ι_1

An m-filtration

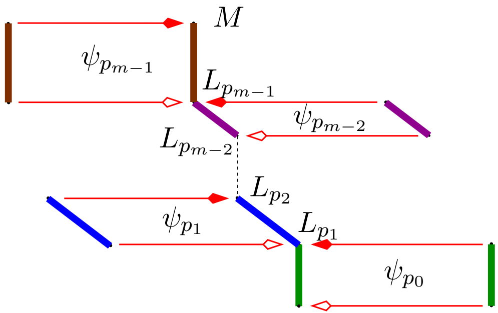

Figure: An ascending m-filtration system

Generalized morphisms

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- Provide a categorical substitute for the annoying diagram chasing of elements which, a priori, do not exist in general ABELian categories.
- Offer an extremely computer-friendly data structure to control the enormous amount of generalized lifting processes needed in homological algebra.

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Thank you for your attention

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