Sheaf and local cohomology

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joint work with
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Overview

1. Coherent sheaves on projective schemes
   - From graded rings to projective schemes
   - From graded modules to quasi-coherent sheaves
2. Sheaf cohomology and the TATE resolution
   - The functor $\mathbf{R}$ and the CASTELNUOVO-MUMFORD regularity
   - The TATE functor $\mathbf{T}$

The Proj construction

Let

$$S = \bigoplus_{i \geq 0} S_i$$

be a graded ring with $S_0 = k$ a field and **maximal homogeneous ideal**

$$\mathfrak{m} := S_{>0} := \bigoplus_{i>0} S_i .$$

Define the scheme

$$X := \operatorname{Proj} S$$

in the following way:

- The underlying set

$$X := \operatorname{Proj} S := \{\mathfrak{p} \lhd S \mid \mathfrak{p} \text{ homogeneous prime and } \mathfrak{p} \not\supset \mathfrak{m}\}.$$

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The Proj construction

- For a *homogeneous* ideal $I \trianglelefteq S$ define the **vanishing locus**

$$V(I) = \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supset I\} = V(\sqrt{I}).$$

- Taking $\{V(I) \mid I \trianglelefteq S \text{ homogeneous}\}$ as the set of closed subsets defines the **Zariski topology** on X.
- For a *homogeneous* $f \in \mathfrak{m}$ define $S_{(f)} := (S_f)_0$.
- The **distinguished open set** $D(f) := \operatorname{Proj} S \setminus V(\langle f \rangle)$.

It follows that

- $D(f) = \operatorname{Spec} S_{(f)}$.
- The distinguished open sets form a basis of the Zariski topology on X.

This suffices to construct the (elementary) **structure sheaf** $\mathcal{O}_X$.

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Projective schemes

Definition

A **projective scheme** $(X, \mathcal{O}_X)$ is a scheme of the form

$$X := \operatorname{Proj} S$$

for some graded ring $S = \bigoplus_{i \geq 0} S_i$.

The quasi-coh. sheaf associated to a graded module

Analogously, for a graded S-module $M_\bullet = \bigoplus_{i \in \mathbb{Z}} M_i$ define the **sheafification**

$$\widetilde{M_\bullet} = \operatorname{Proj} M$$

to be the quasi-coherent (elementary) sheaf on $X = \operatorname{Proj} S$ satisfying

$$(\operatorname{Proj} M)(D(f)) := M_{(f)} := (M_f)_0 := (S_f \otimes_S M)_0$$

for any homogeneous $f \in \mathfrak{m}$.

Quasi-coherent sheaves

Definition

Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **quasi-coherent** if X can be covered by open affine subsets $U_i := \operatorname{Spec} R_i$ with $\mathcal{F}|_{U_i} \cong \operatorname{Spec} M_i$ (where M_i is some R_i-module).

Theorem

Any quasi-coherent sheaf on a projective scheme $X := \operatorname{Proj} S$ is the sheafification

$$\operatorname{Proj} M_\bullet$$

of some graded S-module $M_\bullet$.

Twisting sheaves and twisted sheaves

Define the **twisting**[1] **sheaf** or **twisting line bundle**

$$\mathcal{O}_X(1) = \operatorname{Proj} S(1).$$

More generally define the **twisted line bundles**

$$\mathcal{O}_X(n) = \operatorname{Proj} S(n)$$

for all $n \in \mathbb{Z}$.
For $\mathcal{F} = \operatorname{Proj} M$ define the **twisted sheaves**

$$\mathcal{F}(n) = \operatorname{Proj} M(n) = \operatorname{Proj}(S(n) \otimes_S M) = \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}.$$

[1] Note: The notion of twisting only makes sense in the *projective* context!

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Model quasi-coherent sheaves on graded modules

Good news

We can **model** quasi-coherent sheaves $\mathcal{F}$ on $X = \operatorname{Proj} S$ on graded S-modules $M_\bullet$. Finitely generated graded S-modules give rise to **coherent** sheaves.

Up to ARTINian parts

Bad news

Unfortunately the sheafification does *not* yield an equivalence of categories

$$\{\text{graded } S\text{-modules}\} \xrightarrow[\operatorname{Proj}]{\not\simeq} \{\text{quasi-coh. sheaves on } \operatorname{Proj} S\}$$

Theorem

Two S-modules $M_\bullet$ and $N_\bullet$ define the same quasi-coherent sheaf iff $M_{\geq d} \cong N_{\geq d}$ for some $d \in \mathbb{Z}$, i.e., if they coincide up to ARTINian parts.

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The basic reason for this phenomena is the isomorphism

$$M_{x_i} = (M_{\geq d})_{x_i}.$$

This follows from the **exactness of the localization functor** applied to the short exact sequence

$$0 \to M_{\leq d} \to M \to M/M_{\leq d} \to 0.$$

For a homogeneous element $m \in M/M_{\leq d}$ of degree ℓ we deduce that $m = 1 \cdot m = x_i^{-(d-\ell)} \underbrace{(x_i^{d-\ell} m)}_{=0} = 0$.

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Q:

Is there a way to construct a canonical representative in an equivalence class of graded modules "isomorphic in high degrees"?

A:

One can take the graded S-**module of global sections**

$$\Gamma_\bullet(\operatorname{Proj} M),$$

or any of its truncations, e.g., $\Gamma_{\geq 0}(\operatorname{Proj} M)$.

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Theorem (SERRE-GROTHENDIECK correspondence)

$$\Gamma_\bullet(\operatorname{Proj} M) \cong \varinjlim \operatorname{Hom}_\bullet(\mathfrak{m}^\ell, M)$$

Example (Skyscraper sheaf)

$S := k[x_0, x_1]$, $\mathfrak{m} := \langle x_0, x_1 \rangle$, and $M_\bullet := S/\langle x_1 \rangle$. Check that

$$\operatorname{Hom}_\bullet(\mathfrak{m}^\ell, M) \cong \frac{1}{x_0^\ell} M_\bullet \quad \text{for all } \ell \geq 0.$$

Hence

$$\Gamma_\bullet(\operatorname{Proj} M) \cong (M_{x_0})_\bullet\,,$$

not finitely generated.

Computation won't be efficient in general as it relies on computing $\operatorname{Hom}_\bullet$, i.e., on computing syzygies over S.

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Efficiency?

How to compute the $\mathfrak{m}$-transform efficiently?

Is there a way to compute the $\mathfrak{m}$**-transform**

$$\varinjlim \operatorname{Hom}_\bullet(\mathfrak{m}^\ell, M)$$

in a more efficient way?

Overview

Setting the stage

Let k be a field, V an $n+1$ dimensional k-vector space with basis

$$(e_0, \ldots, e_n),$$

and $W = V^* = \mathrm{Hom}(V, k)$ its k-dual space with **dual basis**

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$E = \bigwedge V$

Define the exterior algebra

$$E = \bigwedge V.$$

Set $\deg e_j = -1$.

$S = \mathrm{Sym}(W)$

Further define the free polynomial ring

$$S := \mathrm{Sym}(W) = k[V] = k[x_0, \ldots, x_n]$$

in $n+1$ indeterminates with $S_0 = k$ and maximal homogeneous ideal $\mathfrak{m} := S_{>0} = \langle x_0, \ldots, x_n \rangle$.

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The R-functor

Idea

The graded S-module structure of $M_\bullet$ can be translated into a complex over the exterior algebra $E := \bigwedge V$.

Take $S := k[x_0, x_1]$ and

$$M_\bullet := S_{\geq 1} = \mathfrak{m} = \langle x_0, x_1 \rangle_S \cong S(-1)^{1\times 2}/(-x_1, x_0).$$

The indeterminates x_0 and x_1 induce maps between

$$M_1 = \langle x_0, x_1 \rangle_k$$

and

$$M_2 = \langle x_0^2, x_0 x_1, x_1^2 \rangle_k$$

given by

$$\mu_0^1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \mu_1^1 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\mathbf{R}(M) : 0 \to E(-1)^2 \xrightarrow{\begin{pmatrix} e_0 & e_1 & 0 \\ 0 & e_0 & e_1 \end{pmatrix}} E(-2)^3 \xrightarrow{\begin{pmatrix} e_0 & e_1 & 0 & 0 \\ 0 & e_0 & e_1 & 0 \\ 0 & 0 & e_0 & e_1 \end{pmatrix}} E(-3)^4 \cdots$$

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The $\mathbf{R}$ functor

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In general we obtain the complex

$$\mathbf{R}(M): \cdots \longrightarrow E(-i) \otimes_k M_i \xrightarrow{\mu^i} E(-i-1) \otimes_k M_{i+1} \xrightarrow{\mu^{i+1}} \cdots,$$

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$$\mu^i := \sum_{j=0}^{n} e_j \mu^i_j$$

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Theorem

- *The functor $\mathbf{R}$ is an equivalence between the category of **graded S-modules** and the category of **linear** free complexes over E.*
- ***Finitely generated** graded S-modules correspond to **left bounded** linear free complexes of E which eventually become exact.*

To construct the f.g. module out of a left bounded free complex set $X = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$. Then

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To construct the f.g. module out of a left bounded free complex set $X = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$. Then

$$
\left(\begin{array}{c|c|c|c|c|c}
X \otimes I_{r_0} & \begin{matrix} -\mu_0^0 \\ \vdots \\ -\mu_n^0 \end{matrix} & 0 & 0 & 0 & 0 \\
\hline
0 & X \otimes I_{r_1} & \begin{matrix} -\mu_0^1 \\ \vdots \\ -\mu_n^1 \end{matrix} & 0 & 0 & 0 \\
\hline
0 & 0 & \ddots & \ddots & 0 & 0 \\
\hline
0 & 0 & 0 & \ddots & \ddots & 0 \\
\hline
0 & 0 & 0 & 0 & X \otimes I_{r_{\mathrm{reg}(M)-1}} & \begin{matrix} -\mu_0^{\mathrm{reg}(M)-1} \\ \vdots \\ -\mu_n^{\mathrm{reg}(M)-1} \end{matrix} \\
\hline
0 & 0 & 0 & 0 & 0 & \mathtt{M}_{\geq \mathrm{reg}\, M}
\end{array}\right)
$$

The following exercise shows how to read off the CASTELNUOVO-MUMFORD regularity of $M_\bullet$ from $\mathbf{R}(M)$.

Exercise

$H^{j-i}(\mathbf{R}(M))_j = \operatorname{Tor}_i^S(k, M)_j$.

Hint: Compute Tor by resolving k.

Corollary

Let $M_\bullet$ be a nontrivial finitely generated graded S-module. Then

$$\operatorname{reg} M := \max\{j - i \mid \beta_{ij} \neq 0\} = \max\{d \mid H^d(\mathbf{R}(M)) \neq 0\}.$$

Proof.

Recall, $\beta_{ij} = \dim_k \operatorname{Tor}_i^S(k, M)_j$. □

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The TATE functor $\mathbf{T}$

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To construct the TATE resolution $\mathbf{T}(M)$ start with the exact complex $\mathbf{R}(M)_{>\operatorname{reg} M}$ and compute an infinite *minimal* free resolution to the left. The TATE resolution only depends on the sheafification of M and we write $\mathbf{T}(\mathcal{F})$ for $\mathcal{F} = \operatorname{Proj} M$.

For $S := k[x_0, x_1]$ and $M_\bullet := S_{\geq 1} = \mathfrak{m}$

$$\mathbf{R}(M_\bullet) : \; 0 \longrightarrow E(-1)^2 \xrightarrow{\begin{pmatrix} e_0 & e_1 & 0 \\ 0 & e_0 & e_1 \end{pmatrix}} \cdots$$

$$\mathbf{T}(M_\bullet) : \cdots \to E(3)^2 \xrightarrow{\begin{pmatrix} e_0 \\ e_1 \end{pmatrix}} E(2)^1 \xrightarrow{\begin{pmatrix} e_0 \cdot e_1 \end{pmatrix}} E(0)^1 \xrightarrow{\begin{pmatrix} e_0 & e_1 \end{pmatrix}} E(-1)^2 \xrightarrow{\begin{pmatrix} e_0 & e_1 & 0 \\ 0 & e_0 & e_1 \end{pmatrix}} \cdots$$

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The BETT diagram

The BETTI diagram for cocomplexes is given by

```
total:   5  4  3  2  1  1  2  3  4  5  6  ?
---------|--|--|--|--|--|--|--|--|--|--|--|
    1:   5  4  3  2  1  .  .  .  .  .  .  0
    0:   *  .  .  .  .  .  1  2  3  4  5  6
---------|--|--|--|--|--|--|--V--|--|--|--|
twist:  -6 -5 -4 -3 -2 -1  0  1  2  3  4  5
-------------------------------------------
Euler:   ? -4 -3 -2 -1  0  1  2  3  4  5  6
```

As we have just seen, the TATE resolution is not a linear complex any more. Killing nonlinearities gives rise to the following definition.

Definition

Let (C, ∂) be *minimal*[a] graded cocomplex of finitely presented graded E-modules. The **linear part** $\operatorname{lin} C$ of C is defined by keeping the objects and erasing all entries in ∂ not having degree -1.

[a] minimal is defined as $\operatorname{im}(\partial) \subseteq \mathfrak{m}C$

Definition

Let T be the TATE resolution of a graded S-module $M_\bullet$. Define the H^i-part of T^m to be the summand of T^m having (internal) degree m, i.e., internal degree equal to i + the cohomological degree. Call it the **i-th linear strand of the TATE resolution $\mathbf{H}^i\mathbf{T}(M)$**

$\mathrm{T}(M)$ and $\mathrm{H}^0_\bullet\mathrm{T}(M)$

Let $S := k[x_0, x_1]$ and $M_\bullet := S^{1\times 2}/\left(x_0 \quad x_0\right)$ with $\mathrm{reg}(M) = 0$:

```
total:    6  5  4  3  2  2  3  4  5  6  7  ?
---------|--|--|--|--|--|--|--|--|--|--|--|
    1:    5  4  3  2  1  .  .  .  .  .  .  0
    0:    *  1  1  1  1  1  2  3  4  5  6  7
---------|--|--|--|--|--|--V--|--|--|--|--|
twist:   -6 -5 -4 -3 -2 -1  0  1  2  3  4  5
-------------------------------------------
Euler:    ? -3 -2 -1  0  1  2  3  4  5  6  7
```

```
    0:    1  1  1  1  1  2  3  4  5  6  7
---------|--|--|--|--|--V--|--|--|--|--|
twist:   -5 -4 -3 -2 -1  0  1  2  3  4  5
----------------------------------------
Euler:    1  1  1  1  1  2  3  4  5  6  7
```

$\mathrm{T}(M)$ and $\mathrm{H}^0_\bullet\mathrm{T}(M)$

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$\mathbf{T}(M)$

$$\cdots \quad E(3) \oplus E(4)^3 \xrightarrow{\begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & e_0 & e_1 \\ 0 & 0 & e_0 \end{pmatrix}} E(2) \oplus E(3)^2 \xrightarrow{\begin{pmatrix} e_1 & 0 \\ 0 & e_1 \\ 0 & e_0 \end{pmatrix}} E(1) \oplus E(2)$$

$$\xrightarrow{\begin{pmatrix} e_1 & e_1 \\ e_0 \cdot e_1 & 0 \end{pmatrix}} E(0)^2 \xrightarrow{\begin{pmatrix} e_1 & -e_0 & 0 \\ 0 & e_0 & e_1 \end{pmatrix}} E(-1)^3 \xrightarrow{\begin{pmatrix} e_1 & 0 & -e_0 & 0 \\ 0 & e_0 & e_1 & 0 \\ 0 & 0 & e_0 & e_1 \end{pmatrix}} E(-2)^4 \quad \cdots$$

$\mathbf{T}(M)$ and $\mathbf{H}^0_\bullet\mathbf{T}(M)$

Let $S := k[x_0, x_1]$ and $M_\bullet := S^{1\times 2} / \begin{pmatrix} x_0 & x_0 \end{pmatrix}$ with $\operatorname{reg}(M) = 0$:

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$$\xrightarrow{\begin{pmatrix} e_1 & e_1 \\ e_0 \cdot e_1 & 0 \end{pmatrix}}$$

$$E(0)^2 \xrightarrow{\begin{pmatrix} e_1 & -e_0 & 0 \\ 0 & e_0 & e_1 \end{pmatrix}} E(-1)^3 \xrightarrow{\begin{pmatrix} e_1 & 0 & -e_0 & 0 \\ 0 & e_0 & e_1 & 0 \\ 0 & 0 & e_0 & e_1 \end{pmatrix}} E(-2)^4 \quad \cdots$$

Theorem ([EFS03, Theorem 4.1])

If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}(W)$ then

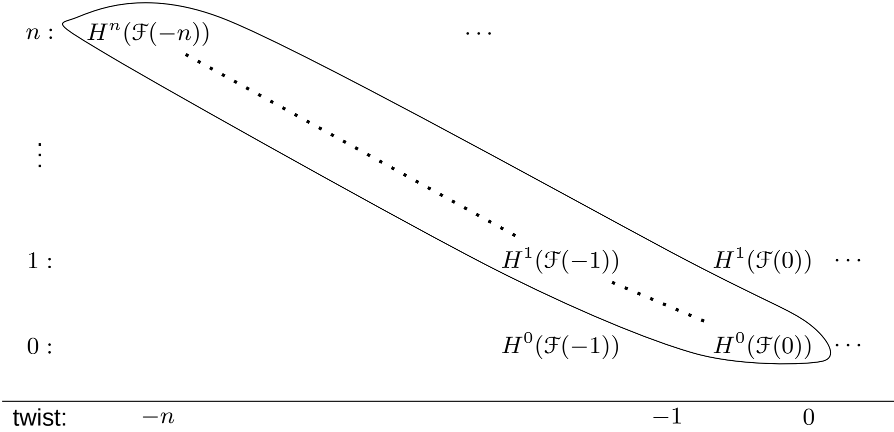
$$\operatorname{lin} \mathbf{T}(\mathcal{F}) = \bigoplus_{i=0}^{n} \mathbf{R}\left(H^i_\bullet(\mathcal{F})\right) = \bigoplus_{i=0}^{n} \mathbf{R}(\bigoplus_{m} H^i(\mathcal{F}(m))).$$

In particular,

$$(\mathbf{T}(\mathcal{F}))^m = \bigoplus_{i} E(-m-i) \otimes_K H^i(\mathcal{F}(m-i)).$$

This yields a method to compute sheaf cohomology.

$(\mathbf{T}(\mathcal{F}))^0 \widehat{=}$



Connection between sheaf and local cohomology

Summary

Let $M_\bullet$ be a graded S-module and $\mathcal{F} := \operatorname{Proj} M$, its sheafification.

1. The linear free E-complex $\mathbf{H}^i\mathbf{T}(M)$ corresponds via the $\mathbf{R}$ functor to then i-th cohomology module

$$H^i_\bullet(\mathcal{F}) := \bigoplus_{d\in\mathbb{Z}} H^i(\mathcal{F}(d)).$$

2. $H^i_\bullet(\mathcal{F}) \cong H^{i+1}_{\mathfrak{m}}(M)$, the $i+1$-st local cohomology of $M_\bullet$.
3. The sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to H^0_\bullet(\mathcal{F}) \to H^1_{\mathfrak{m}}(M) \to 0$$

is exact.

Thank you for your attention

David Eisenbud, Gunnar Fløystad, and Frank-Olaf Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4397–4426 (electronic). MR MR1990756 (2004f:14031)