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SINGULAR Tutorial
Exercise Sheet 1

for beginners

Exercise 1. Start a Singular session.

(1) Define a ring by typing `ring R;`. Then type `R;` or `basering;` to obtain information on the ring R. Observe that R is the polynomial ring $F_{32003}[x, y, z]$ equipped with the degree reverse lexicographic order `dp`. Define the polynomial $f = x^4 + x^3z + x^2y^2 + yz^4 + z^5$. Print f by typing `f;`. How are the monomials of f ordered?
(2) Use the `ring` command to define a new ring S which differs from R only by the choice of the monomial order: choose the lexicographic order `lp`. Use the command `fetch` to map f from R to S (call the "new" polynomial g). Print g. How are the monomials ordered now?

Exercise 2. Define in SINGULAR the polynomial ring in the two variables a, b over the rational numbers with the ordering `dp`. Define in this ring the polynomial

$$f = \frac{33375}{100} \cdot b^6 + a^2 \cdot (11a^2b^2 - b^6 - 121b^4 - 2) + \frac{55}{10} \cdot b^8 + \frac{a}{2 \cdot 33096}$$

and substitute for the variable a the value 77617 and for b the value 33096 using the `subst` command. Now define another ring, the polynomial ring in a, b with real coefficients with precision 10. Repeat everything and compare the results. Now define another ring, the polynomial ring in a, b with real coefficients with precision 20. Repeat everything and compare the results. Repeat this with precision 30.

Exercise 3. Generate 10 homogeneous random polynomials in 5 variables of degree 5 over a finite (!) field (of your choice).

(1) Compute a lexicographic Groebner basis for the ideal generated by the 10 polynomials (use `timer` to check the computing time).
(2) How many Groebner basis elements do you get? Print all elements.
(3) Print the degree of the first and the last element, respectively.
(4) Write the computed Groebner basis to a file named lexGB.out (use the `write` command).
(5) Repeat (1) – (3), replacing the lexicographic order `lp` by the degree reverse lexicographic order `dp`. Write the computed Groebner basis to a file named dpGB.out.

Exercise 4. This exercise is concerned with Singular procedures. Check the SINGULAR on-line help system for `proc`.

(1) Write a Singular procedure which takes as input an ideal I, given by generators $f_1, \ldots, f_r$, and which returns the maximum degree of the f_i.
(2) Apply the procedure to the Groebner bases computed in Exercise 3 (read the data from the files lexGB.out, respectively dpGB.out).

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SINGULAR Tutorial
Exercise Sheet 2

for beginners

Exercise 1. Write a SINGULAR procedure to sort the generators of an ideal by their leading monomials in increasing order.

Apply your procedure to several non-trivial examples of your own choice.

Exercise 2. Implement the normal form algorithm as a SINGULAR procedure.

Exercise 3. Implement Buchberger's algorithm as a SINGULAR procedure.

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SINGULAR Tutorial
Exercise Sheet 3

just for the fun of it

Exercise 1. A well-posed sudoku

	3							
			1	9	5			
	9	8					6	
8				6				
4					3			1
				2				
	6					2	8	
			4	1	9			5
							7	

has a unique solution. In this and further exercises, we will write a SINGULAR library for solving sudokus.

To begin, given a sudoku as above, define

$$E := \{(i,j) | i < j \text{ and } i,j \text{ in the same row, column or } 3 \times 3 - \text{box}\} .$$

Write a SINGULAR procedure which creates E.

Exercise 2. In exercise 1, you numbered all the single squares of a sudoku by $1, \ldots, 81$. In order to assure that your procedures are compatibel to each other, you should keep this numbering throughout all these exercises.

For each $i = 1, \ldots, 81$, let $F_i \in \mathbb{Q}[x_1, \ldots, x_{81}]$ be the polynomial defined by

$$F_i := \prod_{j=1}^{9} (x_i - j) ,$$

and for each pair $(i,j) \in E$, let $G_{i,j} \in \mathbb{Q}[x_1, \ldots, x_{81}]$ be the polynomial defined by

$$G_{i,j} := \frac{F_i - F_j}{x_i - x_j} .$$

Let $F \subset \mathbb{Q}[x_1, \ldots, x_{81}]$ be the ideal generated by the 81 polynomials $\{F_i\}_{i=1,\ldots,81}$ and let $G \subset \mathbb{Q}[x_1, \ldots, x_{81}]$ be the ideal generated by the 810 polynomials $\{G_{i,j}\}_{(i,j)\in E}$. Finally, set $I := F + G \subset \mathbb{Q}[x_1, \ldots, x_{81}]$.

(1) Write a Singular procedure which returns F.
(2) Write a Singular procedure which returns G.

(3) Prove that $a = (a_1, \ldots, a_{81}) \in V(I)$ if and only if $a_i \in \{1, \ldots, 9\}$ and $a_i \neq a_j$ for $(i, j) \in E$.

Exercise 3. Let I be the ideal defined in ex. 12. Let $L \subset \{1, \ldots, 81\}$ be the set of preassigned places and $\{a_i\}_{i \in L}$ the corresponding numbers of a concrete well posed sudoku S. Then

$$I_S := I + < \{x_i - a_i\}_{i \in L} >$$

is the ideal associated to the sudoku S.

Show that the reduced Gröbner basis of I_S with respect to the lexicograhical order has the shape $x_1 - a_1, \ldots, x_{81} - a_{81}$ and $(a_1, \ldots, a_{81})$ is the solution of the sudoku.

Exercise 4. Now put everything together and write a Singular procedure which is capable of solving sudokus where the given sudoku S is represented by a 9×9-`intmat` with zero entries at all places which are not preassigned.

As a suggestion, you could split your solution into three procedures, one for creating the ideal I_S, a second one to read off the solution from the reduced Gröbner basis, and a third one to put everything together.

Use your implementation to solve the sudoku from exercise 1.

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SINGULAR Tutorial
Exercise Sheet 4

for experts

Exercise 1. Write a SINGULAR library which provides functionality to support research on square-free monomial ideals and Stanley-Reisner rings.