Generic Initial Ideals; Lecture 1

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Outline

Zariski open sets and linear automorphisms

Definition of generic initial ideals

Existence

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Zariski open sets

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Lemma 1: Let $U_1, \ldots, U_r \subset K^m$ be nonempty Zariski open sets. Then $U_1 \cap \ldots \cap U_r \neq \emptyset$.

It is enough to show that $U \cap U' \neq \emptyset$, if U and U' are nonempty Zariski open sets of K^m. Let $A = K^m \setminus U$ and $A' = K^m \setminus U'$, and assume that A is the common set of zeroes of the polynomials $f_1, \ldots, f_r$ and A' is the common set of zeroes of the polynomials $g_1, \ldots, g_s$. Let $\mathbf{x} \in U$ and $\mathbf{x}' \in U'$. Then there exist f_i and g_j with $f_i(\mathbf{x}) \neq 0$ and $g_j(\mathbf{x}') \neq 0$.

It follows that $f_i g_j \neq 0$. Since K is infinite, there exists $\mathbf{x}'' \in K^m$ such that $f_i g_j(\mathbf{x}'') \neq 0$. This implies $f_i(\mathbf{x}'') \neq 0$ and $g_j(\mathbf{x}'') \neq 0$. Hence $\mathbf{x}'' \in U \cap U'$.$\checkmark$

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The Lemma implies that a non-empty Zariski open set U in K^n is a dense subsets of K^n. Thus if we choose a "random" point $\mathbf{x} \in K^n$ then it belongs most likely to U.

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Any $\alpha \in \mathrm{GL}_n(K)$, $\alpha = (a_{ij})$ induces an automorphism

$$\alpha : S \to S, \quad f(x_1, \ldots, x_n) \mapsto f(\sum_{i=1}^n a_{i1} x_i, \ldots, \sum_{i=1}^n a_{in} x_i).$$

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This type of automorphism of S is called a linear automorphism.

Definition of generic initial ideals

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Since $\mathrm{GL}_n(K)$ itself is open, a subset of $\mathrm{GL}_n(K)$ is open if and only if it is a Zariski open subset of $K^{n\times n}$.

Theorem 1: Let $I \subset S$ be a graded ideal and $<$ a monomial order on S with $x_1 > x_2 > \cdots > x_n$. Then there exists a nonempty open subset $U \subset \mathrm{GL}_n(K)$ such that

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It is denoted $\mathrm{gin}_<(I)$.

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Given a monomial order $<$ on S, an element $u_1 \wedge u_2 \wedge \cdots \wedge u_t$ where each u_i is a monomial of degree d and where $u_1 > u_2 > \cdots > u_t$, will be called a standard exterior monomial.

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The support of f is the set $\mathrm{supp}(f)$ of standard exterior monomials which appear in f with a nonzero coefficient.

We order the standard exterior monomials lexicographically by setting

$$u_1 \wedge u_2 \wedge \cdots \wedge u_t > v_1 \wedge v_2 \wedge \cdots \wedge v_t,$$

if $u_i > v_i$ for the smallest index i with $u_i \neq v_i$.

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This allows us to define the initial monomial $\mathrm{in}_<(f)$ of a nonzero element $f \in \bigwedge^t S_d$ as the largest standard exterior monomial in the support of f.

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Now let $\alpha \in \mathrm{GL}_n(K)$ be a linear automorphism of S, $V \subset S_d$ a t-dimensional subspace of S_d and $f_1, f_2, \ldots, f_t$ a K-basis of V. Then $\alpha(f_1), \alpha(f_2), \ldots, \alpha(f_t)$ is a K-basis of the vector subspace $\alpha V \subset S_d$.

Lemma 2: Let $w_1 \wedge \cdots \wedge w_t$ be the largest standard exterior monomial of $\bigwedge^t S_d$ with the property that there exists $\alpha \in \mathrm{GL}_n(K)$ with

$$\mathrm{in}_<(\alpha(f_1) \wedge \cdots \wedge \alpha(f_t)) = w_1 \wedge \cdots \wedge w_t.$$

Then the set

$$U = \{\alpha \in \mathrm{GL}_n(K) : \mathrm{in}_<(\alpha(f_1) \wedge \cdots \wedge \alpha(f_t)) = w_1 \wedge \cdots \wedge w_t\}$$

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We observe that if $\mathrm{in}_<(\alpha(f_1) \wedge \cdots \wedge \alpha(f_t)) = w_1 \wedge w_2 \wedge \cdots \wedge w_t$, then $\mathrm{in}_<(\alpha V)$ has the K-basis $w_1, \ldots, w_t$. In particular, $\mathrm{in}_<(\alpha V)$ does not depend on $\alpha \in U$.

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Let $S = K[x_1, x_2]$, and $<$ the lexicographic monomial order on S. Then the standard exterior monomials in $\bigwedge^2 S_2$ are:

$$x_1^2 \wedge x_1x_2 > x_1^2 \wedge x_2^2 > x_1x_2 \wedge x_2^2.$$

Let $f_1 = x_1^2$, $f_2 = x_2^2$ and $\alpha \in \mathrm{GL}_2(K)$. Then

$$\alpha(f_1) = \alpha_{11}^2 x_1^2 + 2\alpha_{11}\alpha_{21}x_1x_2 + \alpha_{21}^2 x_2^2$$

and

$$\alpha(f_2) = \alpha_{12}^2 x_1^2 + 2\alpha_{12}\alpha_{22}x_1x_2 + \alpha_{22}^2 x_2^2$$

. Therefore,

$$\alpha(f_1) \wedge \alpha(f_2) = (2\alpha_{11}^2\alpha_{12}\alpha_{22} - 2\alpha_{12}^2\alpha_{11}\alpha_{21})x_1^2 \wedge x_1x_2 + \cdots,$$

and so $p(\alpha) = 2(\alpha_{11}^2\alpha_{12}\alpha_{22} - \alpha_{12}^2\alpha_{11}\alpha_{21})$.

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Let $d \in \mathbb{Z}_+$ with $I_d \neq 0$. We define the nonempty Zariski open subset $U_d \subset \mathrm{GL}_n(K)$ for the linear subspace $I_d \subset S_d$ similarly to how we defined in Lemma 2 the Zariski open subset $U \subset \mathrm{GL}_n(K)$ for $V \subset S_d$. For those $d \in \mathbb{Z}_+$ with $I_d = 0$, we set $U_d = \mathrm{GL}_n(K)$.

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Let $\alpha \in U_d$ and set $J_d = \mathrm{in}_<(\alpha I_d)$. By the definition of U_d, the vector space J_d does not depend on the particular choice of $\alpha \in U_d$. We claim that $J = \bigoplus_d J_d$ is an ideal.

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In fact, for a given $d \in \mathbb{Z}_+$, we have $U_d \cap U_{d+1} \neq \emptyset$. Then for any $\alpha \in U_d \cap U_{d+1}$ it follows that

$$S_1 J_d = S_1 \,\mathrm{in}_<(\alpha I_d) \subset \mathrm{in}_<(\alpha I_{d+1}) = J_{d+1},$$

which shows that J is indeed an ideal.

Let c be the highest degree of a generator of J, and let $U = U_1 \cap U_2 \cap \cdots \cap U_c$. For any $\alpha \in U$ we will show that $J_d = \mathrm{in}_<(\alpha I_d)$ for all d.

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Now let $d \geq c$. We show by induction on d, that $J_d = \mathrm{in}_<(\alpha I_d)$. For $d = c$, there is nothing to prove. Now let $d > c$. Applying the induction hypothesis we get

$$J_d = S_1 J_{d-1} = S_1 \,\mathrm{in}_<(\alpha I_{d-1}) \subset \mathrm{in}_<(\alpha I_d).$$

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The (nonempty) Zariski open set U just defined, has the desired property. ✓

It can be shown that the Zariski open set $U \subset \mathrm{GL}(n)$ with $\mathrm{gin}_<(I) = \mathrm{in}_<(\alpha I)$ meets non-trivially the group $\mathcal{U}$ of upper triangular matrices with ones on the diagonal. Thus in practice we may choose a “random” $\alpha \in \mathcal{U}$ to compute $\mathrm{gin}_<(I)$ as $\mathrm{in}_<(\alpha I)$.

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We compute $\mathrm{gin}_<(I)$ for $I = (x^2, y^2, z^2)$ and the lexicographic order induced by $x > y > z$. Let

$$\alpha = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

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Then

$$\alpha(I) = (x^2, (ax + y)^2, (bx + cy + z)^2)$$

and we get

$$\text{gin}_<(I) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4), \quad \text{if}$$

$$\text{char}\, K \neq 2, 3 \quad \text{and} \quad abc(ac - b) \neq 0,$$

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One might ask whether the gin of a complete intersection does depend on the specific complete intersection. Not surprisingly it does.

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With CoCoA we get:

$$\operatorname{gin}(a^3, b^3, c^3, d^3) = (a^3, a^2b, ab^2, b^3, a^2c^2, abc^2, ac^3, bc^4, c^5, b^2c^3, b^2c^2d, bc^3d, c^4d, a^2cd^3, abcd^3, b^2cd^3, ac^2d^3, bc^2d^3, c^3d^3, a^2d^5, b^2d^5, abd^5, acd^5, bcd^5, c^2d^5, ad^7, bd^7, cd^7, d^9),$$

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$$\text{gin}(a^3, b^3, c^3, d^3) = (a^3, a^2b, ab^2, b^3, a^2c^2, abc^2, ac^3, bc^4, c^5, b^2c^3, b^2c^2d, bc^3d, c^4d, a^2cd^3, abcd^3, b^2cd^3, ac^2d^3, bc^2d^3, c^3d^3, a^2d^5, b^2d^5, abd^5, acd^5, bcd^5, c^2d^5, ad^7, bd^7, cd^7, d^9),$$

while for a generic complete intersection I we have

$$\text{gin}(I) = (a^3, a^2b, ab^2, b^3, a^2c^2, abc^2, b^2c^2, ac^4, bc^4, c^5, ac^3d, bc^3d, c^4d, a^2cd^3, abcd^3, b^2cd^3, ac^2d^3, bc^2d^3, c^3d^3, a^2d^5, abd^5, b^2d^5, acd^5, bcd^5, c^2d^5, ad^7, bd^7, cd^7, d^9)$$

The number of generators of the generic initial ideal of a 0-dimensional generic complete intersection in $K[x_1, \ldots, x_n]$ generated in degree d.

$n \rightarrow$

$d \downarrow$	2	3	4	5
2	3	6	12	21
3	4	11	29	76
4	5	17	60	206
5	6	25	108	473

The same diagram for $\mathrm{gin}_<(x_1^d, \ldots, x_n^d)$.

$n \rightarrow$

$d \downarrow$		2	3	4	5
	2	3	6	12	21
	3	4	11	29	77
	4	5	17	60	207
	5	6	25	108	474

The same diagram for $\mathrm{gin}_<(x_1^d, \ldots, x_n^d)$.

$n \rightarrow$

$d \downarrow$	2	3	4	5
2	3	6	12	21
3	4	11	29	77
4	5	17	60	207
5	6	25	108	474

Problem: Give an explicit formula for the minimal number of generators $\mu(n, d)$ of $\mathrm{gin}_<(x_1^d, \ldots, x_n^d)$.

I asked the Encyclopedia of Integer Sequences

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Interpretation of the second column:
Index of 5^n within the sequence of numbers of form $3^i 5^j$.

$$1, 3, 5, 9, 15, 25, 27, 45, 75, 81, 125, \cdots$$

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$$1, 3, 5, 9, 15, 25, 27, 45, 75, 81, 125, \cdots$$

Interpretation of the third column:

I'm sorry, but your terms do not match anything in the table.