Generic Initial Ideals; Lecture 2

Jürgen Herzog
Universität Duisburg-Essen

Workshop on Computational Commutative Algebra,

July 2–7, 2011

University of Tehran and IPM

Outline

The theorem of Galligo and Bayer–Stillman

Strongly stable and Borel fixed ideals

Ideals of Borel type

Outline

The theorem of Galligo and Bayer–Stillman

Strongly stable and Borel fixed ideals

Ideals of Borel type

Outline

The theorem of Galligo and Bayer–Stillman

The subgroup $\mathcal{B} \subset \mathrm{GL}_n(K)$ of all nonsingular upper triangular matrices is called the Borel subgroup of $\mathrm{GL}_n(K)$.

The theorem of Galligo and Bayer–Stillman

The subgroup $\mathcal{B} \subset \mathrm{GL}_n(K)$ of all nonsingular upper triangular matrices is called the Borel subgroup of $\mathrm{GL}_n(K)$.

A matrix $\alpha = (a_{ij}) \in \mathcal{B}$ is called an upper elementary matrix, if $a_{ii} = 1$ for all i and if there exist integers$1 \leq k < l \leq n$ such that $a_{kl} \neq 0$ while $a_{ij} = 0$ for all $i \neq j$ with $\{i, j\} \neq \{k, l\}$.

The theorem of Galligo and Bayer–Stillman

The subgroup $\mathcal{B} \subset \mathrm{GL}_n(K)$ of all nonsingular upper triangular matrices is called the Borel subgroup of $\mathrm{GL}_n(K)$.

A matrix $\alpha = (a_{ij}) \in \mathcal{B}$ is called an upper elementary matrix, if $a_{ii} = 1$ for all i and if there exist integers$1 \leq k < l \leq n$ such that $a_{kl} \neq 0$ while $a_{ij} = 0$ for all $i \neq j$ with $\{i, j\} \neq \{k, l\}$.

Recall from linear algebra that the subgroup $\mathcal{D} \subset \mathcal{B}$ of all nonsingular diagonal matrices together with the set of all upper elementary matrices generate $\mathcal{B}$.

The following theorem gives us information about generic initial ideals

The following theorem gives us information about generic initial ideals

Theorem 1:(Galligo, Bayer–Stillman) Let $I \subset S$ be a graded ideal and $<$ a monomial order on S. Then $\text{gin}_<(I)$ is a Borel-fixed ideal, that is, $\alpha(\text{gin}_<(I)) = \text{gin}_<(I)$ for all $\alpha \in \mathcal{B}$.

The following theorem gives us information about generic initial ideals

Theorem 1:(Galligo, Bayer–Stillman) Let $I \subset S$ be a graded ideal and $<$ a monomial order on S. Then $\text{gin}_<(I)$ is a Borel-fixed ideal, that is, $\alpha(\text{gin}_<(I)) = \text{gin}_<(I)$ for all $\alpha \in \mathcal{B}$.

Reduction to the upper triangular elementary matrices: Notice that an invertible diagonal matrix δ keeps monomial ideals fixed, because if $d_1, \ldots, d_n$ is the diagonal of δ and u is a monomial, then $\delta(u) = u(d_1, \ldots, d_n)u$ and $u(d_1, \ldots, d_n) \in K \setminus \{0\}$. Here $u(d_1, \ldots, d_n)$ denotes the evaluation of the monomial u at the point $(d_1, \ldots, d_n)$, that is, if $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, then $u(d_1, \ldots, d_n) = d_1^{a_1} d_2^{a_2} \cdots d_n^{a_n}$.

The following theorem gives us information about generic initial ideals

Theorem 1:(Galligo, Bayer–Stillman) Let $I \subset S$ be a graded ideal and $<$ a monomial order on S. Then $\text{gin}_<(I)$ is a Borel-fixed ideal, that is, $\alpha(\text{gin}_<(I)) = \text{gin}_<(I)$ for all $\alpha \in \mathcal{B}$.

Reduction to the upper triangular elementary matrices: Notice that an invertible diagonal matrix δ keeps monomial ideals fixed, because if $d_1, \ldots, d_n$ is the diagonal of δ and u is a monomial, then $\delta(u) = u(d_1, \ldots, d_n)u$ and $u(d_1, \ldots, d_n) \in K \setminus \{0\}$. Here $u(d_1, \ldots, d_n)$ denotes the evaluation of the monomial u at the point $(d_1, \ldots, d_n)$, that is, if $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, then $u(d_1, \ldots, d_n) = d_1^{a_1} d_2^{a_2} \cdots d_n^{a_n}$.

Suppose now there is an element $\alpha \in \mathcal{B}$ with $\alpha(\text{gin}_<(I)) \neq \text{gin}_<(I)$. Then, since $\mathcal{B}$ is generated by invertible diagonal matrices (which fix $\text{gin}_<(I)$) and by upper elementary matrices, we may assume that α is an upper elementary matrix. This leads to a contradiction. $\checkmark$

Strongly stable and Borel fixed ideals

Definition: Let $I \subset S$ be a monomial ideal. Then I is called strongly stable if one has $x_i(u/x_j) \in I$ for all monomials $u \in I$ and all $i < j$ such that x_j divides u.

Strongly stable and Borel fixed ideals

Definition: Let $I \subset S$ be a monomial ideal. Then I is called strongly stable if one has $x_i(u/x_j) \in I$ for all monomials $u \in I$ and all $i < j$ such that x_j divides u.

It is enough to check the defining property of strongly stable ideals on the generators of I.

Strongly stable and Borel fixed ideals

Definition: Let $I \subset S$ be a monomial ideal. Then I is called strongly stable if one has $x_i(u/x_j) \in I$ for all monomials $u \in I$ and all $i < j$ such that x_j divides u.

It is enough to check the defining property of strongly stable ideals on the generators of I.

Why are we interested in strongly stable ideals?

Strongly stable and Borel fixed ideals

Definition: Let $I \subset S$ be a monomial ideal. Then I is called strongly stable if one has $x_i(u/x_j) \in I$ for all monomials $u \in I$ and all $i < j$ such that x_j divides u.

It is enough to check the defining property of strongly stable ideals on the generators of I.

Why are we interested in strongly stable ideals?

Here is the answer:

Proposition 1: (a) Let $I \subset S$ be a graded ideal. Then I is a monomial ideal, if I is Borel-fixed.

Strongly stable and Borel fixed ideals

Definition: Let $I \subset S$ be a monomial ideal. Then I is called strongly stable if one has $x_i(u/x_j) \in I$ for all monomials $u \in I$ and all $i < j$ such that x_j divides u.

It is enough to check the defining property of strongly stable ideals on the generators of I.

Why are we interested in strongly stable ideals?

Here is the answer:

Proposition 1: (a) Let $I \subset S$ be a graded ideal. Then I is a monomial ideal, if I is Borel-fixed.

(b) Let I be a Borel-fixed ideal and a the largest exponent appearing among the monomial generators of I. If $\operatorname{char} K = 0$ or $\operatorname{char} K > a$, then I is strongly stable.

Strongly stable and Borel fixed ideals

Definition: Let $I \subset S$ be a monomial ideal. Then I is called strongly stable if one has $x_i(u/x_j) \in I$ for all monomials $u \in I$ and all $i < j$ such that x_j divides u.

It is enough to check the defining property of strongly stable ideals on the generators of I.

Why are we interested in strongly stable ideals?

Here is the answer:

Proposition 1: (a) Let $I \subset S$ be a graded ideal. Then I is a monomial ideal, if I is Borel-fixed.

(b) Let I be a Borel-fixed ideal and a the largest exponent appearing among the monomial generators of I. If $\operatorname{char} K = 0$ or $\operatorname{char} K > a$, then I is strongly stable.

(c) If I is strongly stable, then I is Borel-fixed.

Proof (a) We show that if $f \in I$ is a nonzero homogeneous polynomial, and $u \in \text{supp}(f)$, then there exists a homogeneous polynomial $g \in I$ with $\text{supp}(g) = \text{supp}(f) \setminus \{u\}$.

Proof (a) We show that if $f \in I$ is a nonzero homogeneous polynomial, and $u \in \text{supp}(f)$, then there exists a homogeneous polynomial $g \in I$ with $\text{supp}(g) = \text{supp}(f) \setminus \{u\}$.

Suppose $f = a_u u + \sum_{v \neq u} a_v v$, and $\alpha \in \mathcal{B}$ is a diagonal matrix with diagonal $c_1, c_2, \ldots, c_n$. Then
$\alpha(f) = a_u u(c_1, \ldots, c_n) u + \sum_{v \neq u} a_v v(c_1, \ldots, c_n) v$.

Proof (a) We show that if $f \in I$ is a nonzero homogeneous polynomial, and $u \in \text{supp}(f)$, then there exists a homogeneous polynomial $g \in I$ with $\text{supp}(g) = \text{supp}(f) \setminus \{u\}$.

Suppose $f = a_u u + \sum_{v \neq u} a_v v$, and $\alpha \in \mathcal{B}$ is a diagonal matrix with diagonal $c_1, c_2, \dots, c_n$. Then
$\alpha(f) = a_u u(c_1, \dots, c_n)u + \sum_{v \neq u} a_v v(c_1, \dots, c_n)v$.

Since K is infinite, we may choose $c_1, \dots, c_n$ such that
$u(c_1, \dots, c_n) \neq v(c_1, \dots, c_n)$ for all $v \neq u$. Let
$g = u(c_1, \dots, c_n)f - \alpha(f)$. Then, indeed, we have
$\text{supp}(g) = \text{supp}(f) \setminus \{u\}$.

(b) Let $u \in I$ be a monomial, x_j a variable which divides u and $1 \leq i < j$ a number.

(b) Let $u \in I$ be a monomial, x_j a variable which divides u and $1 \leq i < j$ a number.

Let $\alpha \in \mathcal{B}$ be the upper elementary matrix which induces the linear automorphism on S with $x_k \mapsto x_k$ for $k \neq j$ and $x_j \mapsto x_i + x_j$.

(b) Let $u \in I$ be a monomial, x_j a variable which divides u and $1 \leq i < j$ a number.

Let $\alpha \in \mathcal{B}$ be the upper elementary matrix which induces the linear automorphism on S with $x_k \mapsto x_k$ for $k \neq j$ and $x_j \mapsto x_i + x_j$.

Suppose that $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$; then

$$\alpha(u) = x_1^{a_1} x_2^{a_2} \cdots (x_i + x_j)^{a_j} \cdots x_n^{a_n} = u + a_j x_i (u/x_j) + \cdots .$$

Since I is Borel-fixed, it follows that $\alpha(u) \in I$, and since I is a monomial ideal, we have $\text{supp}(\alpha(u)) \subset I$. The assumption on the characteristic on K and the above calculation then shows that $x_i(u/x_j) \in I$. ✓

Let $u, v \in S_d$ be two monomials, $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ and $v = x_{j_1} x_{j_2} \cdots x_{j_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ and $j_1 \leq j_2 \leq \cdots \leq j_d$. We define the partial order: $u \succeq v \Leftrightarrow i_k \leq j_k$ for all k.

Let $u, v \in S_d$ be two monomials, $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ and $v = x_{j_1} x_{j_2} \cdots x_{j_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ and $j_1 \leq j_2 \leq \cdots \leq j_d$. We define the partial order: $u \succeq v \Leftrightarrow i_k \leq j_k$ for all k.

This order is called Borel order. A monomial ideal I is strongly stable, if for all d, and all monomials $v \in I_d$ and $u \in S_d$ with $u \succeq v$, one has that $u \in I_d$.

Let $u, v \in S_d$ be two monomials, $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ and $v = x_{j_1}x_{j_2}\cdots x_{j_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ and $j_1 \leq j_2 \leq \cdots \leq j_d$. We define the partial order: $u \succeq v \Leftrightarrow i_k \leq j_k$ for all k.

This order is called Borel order. A monomial ideal I is strongly stable, if for all d, and all monomials $v \in I_d$ and $u \in S_d$ with $u \succeq v$, one has that $u \in I_d$.

Moreover, if $u \succeq v$, then $u > v$ with respect to any monomial order $>$ on S with $x_1 > x_2 > \cdots > x_n$.

Let $u, v \in S_d$ be two monomials, $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ and $v = x_{j_1}x_{j_2}\cdots x_{j_d}$ with $i_1 \le i_2 \le \cdots \le i_d$ and $j_1 \le j_2 \le \cdots \le j_d$. We define the partial order: $u \succeq v \Leftrightarrow i_k \le j_k$ for all k.

This order is called Borel order. A monomial ideal I is strongly stable, if for all d, and all monomials $v \in I_d$ and $u \in S_d$ with $u \succeq v$, one has that $u \in I_d$.

Moreover, if $u \succeq v$, then $u > v$ with respect to any monomial order $>$ on S with $x_1 > x_2 > \cdots > x_n$.

Proposition 2: Let $I \subset S$ be a graded ideal and $<$ a monomial order on S. Then:

Let $u, v \in S_d$ be two monomials, $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ and $v = x_{j_1} x_{j_2} \cdots x_{j_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ and $j_1 \leq j_2 \leq \cdots \leq j_d$. We define the partial order: $u \succeq v \Leftrightarrow i_k \leq j_k$ for all k.

This order is called Borel order. A monomial ideal I is strongly stable, if for all d, and all monomials $v \in I_d$ and $u \in S_d$ with $u \succeq v$, one has that $u \in I_d$.

Moreover, if $u \succeq v$, then $u > v$ with respect to any monomial order $>$ on S with $x_1 > x_2 > \cdots > x_n$.

Proposition 2: Let $I \subset S$ be a graded ideal and $<$ a monomial order on S. Then:

(a) $\operatorname{gin}_<(I)$ is strongly stable, if $\operatorname{char} K = 0$.

Let $u, v \in S_d$ be two monomials, $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ and $v = x_{j_1} x_{j_2} \cdots x_{j_d}$ with $i_1 \le i_2 \le \cdots \le i_d$ and $j_1 \le j_2 \le \cdots \le j_d$. We define the partial order: $u \succeq v \Leftrightarrow i_k \le j_k$ for all k.

This order is called Borel order. A monomial ideal I is strongly stable, if for all d, and all monomials $v \in I_d$ and $u \in S_d$ with $u \succeq v$, one has that $u \in I_d$.

Moreover, if $u \succeq v$, then $u > v$ with respect to any monomial order $>$ on S with $x_1 > x_2 > \cdots > x_n$.

Proposition 2: Let $I \subset S$ be a graded ideal and $<$ a monomial order on S. Then:

(a) $\mathrm{gin}_<(I)$ is strongly stable, if $\mathrm{char}\, K = 0$.

(b) (Conca) $\mathrm{gin}_<(I) = I$ if and only if I is Borel-fixed.

Proof: (a) follows from Theorem 1 and Proposition 1.

Proof: (a) follows from Theorem 1 and Proposition 1.

(b) One direction of the assertion is a consequence of Theorem 1.

Proof: (a) follows from Theorem 1 and Proposition 1.

(b) One direction of the assertion is a consequence of Theorem 1.

For the other direction we use the fact that a matrix α whose principal minors are all nonzero, can be written as a product $\beta\gamma$ where β is an invertible lower triangular matrix and γ an invertible upper triangular matrix. This is an open condition. Thus we may choose $\alpha \in \mathrm{GL}_n(K)$ with $\mathrm{gin}_<(I) = \mathrm{in}_<(\alpha I)$ and which has a product presentation $\alpha = \beta\gamma$, as described above.

Proof: (a) follows from Theorem 1 and Proposition 1.

(b) One direction of the assertion is a consequence of Theorem 1.

For the other direction we use the fact that a matrix α whose principal minors are all nonzero, can be written as a product $\beta\gamma$ where β is an invertible lower triangular matrix and γ an invertible upper triangular matrix. This is an open condition. Thus we may choose $\alpha \in \mathrm{GL}_n(K)$ with $\mathrm{gin}_<(I) = \mathrm{in}_<(\alpha I)$ and which has a product presentation $\alpha = \beta\gamma$, as described above.

For the invertible lower triangular matrix β and any monomial u one has $\beta(u) = au + \cdots$ with $a \in K \setminus \{0\}$ and with $u \succeq v$ for all $v \in \mathrm{supp}(\beta(u))$ such that $v \neq u$.

Proof: (a) follows from Theorem 1 and Proposition 1.

(b) One direction of the assertion is a consequence of Theorem 1.

For the other direction we use the fact that a matrix α whose principal minors are all nonzero, can be written as a product $\beta\gamma$ where β is an invertible lower triangular matrix and γ an invertible upper triangular matrix. This is an open condition. Thus we may choose $\alpha \in \mathrm{GL}_n(K)$ with $\mathrm{gin}_<(I) = \mathrm{in}_<(\alpha I)$ and which has a product presentation $\alpha = \beta\gamma$, as described above.

For the invertible lower triangular matrix β and any monomial u one has $\beta(u) = au + \cdots$ with $a \in K \setminus \{0\}$ and with $u \succeq v$ for all $v \in \mathrm{supp}(\beta(u))$ such that $v \neq u$.

It follows therefore that for every homogeneous polynomial f, one has $\mathrm{in}_<(\beta(f)) = \mathrm{in}_<(f)$. This implies that $\mathrm{in}_<(\beta I) = \mathrm{in}_<(I)$.

Therefore,

$$\text{gin}_<(I) = \text{in}_<(\beta\gamma I) = \text{in}_<(\gamma I) = \text{in}_<(I) = I.$$

Here we used that $\gamma I = I$, since by assumption, I is Borel-fixed. The last equation holds, since I is a monomial ideal. ✓

Therefore,

$$\text{gin}_<(I) = \text{in}_<(\beta\gamma I) = \text{in}_<(\gamma I) = \text{in}_<(I) = I.$$

Here we used that $\gamma I = I$, since by assumption, I is Borel-fixed. The last equation holds, since I is a monomial ideal. ✓

Corollary 1: Let $I \subset S$ be a graded ideal and $<$ a monomial order on S. Then $\text{gin}_<(\text{gin}_<(I)) = \text{gin}_<(I)$.

Ideals of Borel type

Definition: A monomial ideal $I \subset S = K[x_1, \ldots, x_n]$ is of Borel type if

$$I : x_i^\infty = I : (x_1, \ldots, x_i)^\infty \quad \text{for} \quad i = 1, \ldots, n.$$

Ideals of Borel type

Definition: A monomial ideal $I \subset S = K[x_1, \ldots, x_n]$ is of Borel type if

$$I : x_i^\infty = I : (x_1, \ldots, x_i)^\infty \quad \text{for} \quad i = 1, \ldots, n.$$

Theorem 2: (Bayer–Stillman) Borel-fixed ideals are of Borel type.

Ideals of Borel type

Definition: A monomial ideal $I \subset S = K[x_1, \ldots, x_n]$ is of Borel type if

$$I : x_i^\infty = I : (x_1, \ldots, x_i)^\infty \quad \text{for} \quad i = 1, \ldots, n.$$

Theorem 2: (Bayer–Stillman) Borel-fixed ideals are of Borel type.

For the proof of the theorem we need the following characterization of Borel-fixed ideals.

Proposition 3: Let $I \subset S$ be a monomial ideal. The following conditions are equivalent:

Proposition 3: Let $I \subset S$ be a monomial ideal. The following conditions are equivalent:

(a) I is of Borel type.

Proposition 3: Let $I \subset S$ be a monomial ideal. The following conditions are equivalent:

(a) I is of Borel type.

(b) For each monomial $u \in I$ and all integers i, j, s with $1 \leq j < i \leq n$ and $s > 0$ such that $x_i^s | u$ there exists an integer $t \geq 0$ such that $x_j{}^t (u/x_i^s) \in I$.

Proposition 3: Let $I \subset S$ be a monomial ideal. The following conditions are equivalent:

(a) I is of Borel type.

(b) For each monomial $u \in I$ and all integers i, j, s with $1 \leq j < i \leq n$ and $s > 0$ such that $x_i^s | u$ there exists an integer $t \geq 0$ such that $x_j{}^t(u/x_i^s) \in I$.

(c) For each monomial $u \in I$ and all integers i, j with $1 \leq j < i \leq n$ there exists an integer $t \geq 0$ such that $x_j{}^t(u/x_i^{\nu_i(u)}) \in I$.

Proposition 3: Let $I \subset S$ be a monomial ideal. The following conditions are equivalent:

(a) I is of Borel type.

(b) For each monomial $u \in I$ and all integers i, j, s with $1 \leq j < i \leq n$ and $s > 0$ such that $x_i^s | u$ there exists an integer $t \geq 0$ such that $x_j{}^t(u/x_i^s) \in I$.

(c) For each monomial $u \in I$ and all integers i, j with $1 \leq j < i \leq n$ there exists an integer $t \geq 0$ such that $x_j{}^t(u/x_i^{\nu_i(u)}) \in I$.

(d) If $P \in \mathrm{Ass}(S/I)$, then $P = (x_1, \ldots, x_j)$ for some j.

Proof: (a) $\Rightarrow$ (b): Let $u \in I$ be a monomial such that $x_i^s | u$ for some $s > 0$, and let $j < i$. Then $u = x_i^s v$ with $v \in I : x_i^\infty$. Condition (a) implies that $I : x_i^\infty \subset I : x_j^\infty$. Therefore, there exists an integer $t \geq 0$ such that $x_j^t(u/x_i^s) = x_j^t v \in I$.

Proof: (a) $\Rightarrow$ (b): Let $u \in I$ be a monomial such that $x_i^s | u$ for some $s > 0$, and let $j < i$. Then $u = x_i^s v$ with $v \in I : x_i^\infty$. Condition (a) implies that $I : x_i^\infty \subset I : x_j^\infty$. Therefore, there exists an integer $t \geq 0$ such that $x_j^t (u / x_i^s) = x_j^t v \in I$.

(b) $\Rightarrow$ (c) it trivial.

Proof: (a) $\Rightarrow$ (b): Let $u \in I$ be a monomial such that $x_i^s | u$ for some $s > 0$, and let $j < i$. Then $u = x_i^s v$ with $v \in I : x_i^\infty$. Condition (a) implies that $I : x_i^\infty \subset I : x_j^\infty$. Therefore, there exists an integer $t \geq 0$ such that $x_j^t (u/x_i^s) = x_j^t v \in I$.

(b) $\Rightarrow$ (c) it trivial.

Proof of Theorem 2: We know that I is a monomial ideal. We will show that I satisfies condition (c) of Proposition 3.

Proof: (a) $\Rightarrow$ (b): Let $u \in I$ be a monomial such that $x_i^s | u$ for some $s > 0$, and let $j < i$. Then $u = x_i^s v$ with $v \in I : x_i^\infty$. Condition (a) implies that $I : x_i^\infty \subset I : x_j^\infty$. Therefore, there exists an integer $t \geq 0$ such that $x_j^t(u/x_i^s) = x_j^t v \in I$.

(b) $\Rightarrow$ (c) it trivial.

Proof of Theorem 2: We know that I is a monomial ideal. We will show that I satisfies condition (c) of Proposition 3.

Let $u \in I$ with $a = \nu_i(u)$, and let $1 \leq j < i$. We want to find an integer t such that $x_j^t(u/x_i^a) \in I$. If $a = 0$, there is nothing to show.

Proof: (a) $\Rightarrow$ (b): Let $u \in I$ be a monomial such that $x_i^s | u$ for some $s > 0$, and let $j < i$. Then $u = x_i^s v$ with $v \in I : x_i^\infty$. Condition (a) implies that $I : x_i^\infty \subset I : x_j^\infty$. Therefore, there exists an integer $t \geq 0$ such that $x_j^t (u/x_i^s) = x_j^t v \in I$.

(b) $\Rightarrow$ (c) it trivial.

Proof of Theorem 2: We know that I is a monomial ideal. We will show that I satisfies condition (c) of Proposition 3.

Let $u \in I$ with $a = \nu_i(u)$, and let $1 \leq j < i$. We want to find an integer t such that $x_j^t (u/x_i^a) \in I$. If $a = 0$, there is nothing to show.

Suppose now that $a > 0$. Since I is Borel-fixed, the polynomial $\sum_{k=0}^a \binom{a}{k} x_j^k (u/x_i^k)$ belongs to I ($x_i \mapsto x_j + x_i$). Thus, since I is a monomial ideal, it follows that $x_j^k (u/x_i^k) \in I$ for all k with $\binom{a}{k} \neq 0$ in K. Hence if $\operatorname{char} K = 0$, then $x_j^k (u/x_i^k) \in I$ for all $k = 0, \ldots, a$.

Now assume that $\mathrm{char}\, K = p > 0$, and let $a = \sum_i a_i p^i$ be the p-adic expansion of a. Let j be an index such that $a_j \neq 0$, and let $k = p^j$. Then $\binom{a}{k} = a_j \neq 0$ in K. This follows from the following identity

$$\binom{a}{k} = \prod_i \binom{a_i}{k_i} \bmod p,$$

of Lucas, where $k = \sum_i k_i p^i$ is the p-adic expansion of k.

Now assume that $\operatorname{char} K = p > 0$, and let $a = \sum_i a_i p^i$ be the p-adic expansion of a. Let j be an index such that $a_j \neq 0$, and let $k = p^j$. Then $\binom{a}{k} = a_j \neq 0$ in K. This follows from the following identity

$$\binom{a}{k} = \prod_i \binom{a_i}{k_i} \bmod p,$$

of Lucas, where $k = \sum_i k_i p^i$ is the p-adic expansion of k.

Therefore in all cases there exists an integer k with $1 \leq k \leq a$ such that $x_j^k(u/x_i^k) \in I$. Set $a' = a - k$ and $u' = x_j^k(u/x_i^k)$. Then $\nu_i(u') = a' < a$.

Now assume that $\operatorname{char} K = p > 0$, and let $a = \sum_i a_i p^i$ be the p-adic expansion of a. Let j be an index such that $a_j \neq 0$, and let $k = p^j$. Then $\binom{a}{k} = a_j \neq 0$ in K. This follows from the following identity

$$\binom{a}{k} = \prod_i \binom{a_i}{k_i} \bmod p,$$

of Lucas, where $k = \sum_i k_i p^i$ is the p-adic expansion of k.

Therefore in all cases there exists an integer k with $1 \leq k \leq a$ such that $x_j^k(u/x_i^k) \in I$. Set $a' = a - k$ and $u' = x_j^k(u/x_i^k)$. Then $\nu_i(u') = a' < a$.

Arguing by induction on a, we may assume that there exists an integer t' such that $x_j^{t'}(u'/x_i^{a'}) \in I$. Thus if set $t = t' + k$, then $x_j^t(u/x_i^a) \in I$, as desired. ✓

A more precise statement concerning Borel fixed ideals is due to Pardue.

A more precise statement concerning Borel fixed ideals is due to Pardue.

Let K be a field of characteristic > 0. Write $x_i^l || u$ to express that x_i^l divides u but x_i^{l+1} does not, and for non-negative integers k and l with p-adic expansion $k = \sum_i k_i p^i$ and $l = \sum_i l_i p^i$, set $k \leq_p l$ if $k_i \leq l_i$ for all i.

A more precise statement concerning Borel fixed ideals is due to Pardue.

Let K be a field of characteristic > 0. Write $x_i^l || u$ to express that x_i^l divides u but x_i^{l+1} does not, and for non-negative integers k and l with p-adic expansion $k = \sum_i k_i p^i$ and $l = \sum_i l_i p^i$, set $k \leq_p l$ if $k_i \leq l_i$ for all i.

Then a monomial ideal I is Borel-fixed if and only if it satisfies the following condition: if u is a monomial in I and $x_j^l || u$, then $(x_i/x_j)^k u \in I$ for all $i < j$, and all $k \leq_p l$.

A more precise statement concerning Borel fixed ideals is due to Pardue.

Let K be a field of characteristic > 0. Write $x_i^l || u$ to express that x_i^l divides u but x_i^{l+1} does not, and for non-negative integers k and l with p-adic expansion $k = \sum_i k_i p^i$ and $l = \sum_i l_i p^i$, set $k \leq_p l$ if $k_i \leq l_i$ for all i.

Then a monomial ideal I is Borel-fixed if and only if it satisfies the following condition: if u is a monomial in I and $x_j^l || u$, then $(x_i/x_j)^k u \in I$ for all $i < j$, and all $k \leq_p l$.

Pardue calls a monomial ideal satisfying this combinatorial condition p-Borel, regardless of the characteristic of K.

While the homological properties of strongly stable ideals are well understood, as we shall see in later lectures, those of general Borel type are difficult to understand.

While the homological properties of strongly stable ideals are well understood, as we shall see in later lectures, those of general Borel type are difficult to understand.

Among the p-Borel ideals the principal ones are the most simple. Let u be a monomial; then $\langle u \rangle$ denotes the smallest p-Borel ideal which contains u. The ideal $\langle u \rangle$ is called principal p-Borel with Borel generator u.

While the homological properties of strongly stable ideals are well understood, as we shall see in later lectures, those of general Borel type are difficult to understand.

Among the p-Borel ideals the principal ones are the most simple. Let u be a monomial; then $\langle u\rangle$ denotes the smallest p-Borel ideal which contains u. The ideal $\langle u\rangle$ is called principal p-Borel with Borel generator u.

Proposition 4: (Pardue) Let $u = \prod_{i=1}^n x_i^{\mu_i}$, and let $\mu_i = \sum_j \mu_{ij} p^j$ for $i = 1, \ldots, n$ be the p-adic expansion of the exponents of u. Then

$$\langle u\rangle = \prod_{i=1}^{n} \prod_{j} ((x_1, \ldots, x_j)^{\mu_{ij}})^{[p^j]}.$$

In particular, $\langle u\rangle = \prod_{i=1}^n \langle x_i^{\mu_i}\rangle$.

What is the regularity and what the projective dimension of $\langle u \rangle$?

What is the regularity and what the projective dimension of $\langle u \rangle$?

Pardue showed that $\mathrm{reg}\langle x^{\mu} \rangle = \mu_1 + \mathrm{reg}\langle \frac{x^{\mu}}{x_1^{\mu_1}} \rangle$, therefore we can assume that x_1 does not divide x^{μ}.

What is the regularity and what the projective dimension of $\langle u \rangle$?

Pardue showed that $\text{reg}\langle x^{\mu} \rangle = \mu_1 + \text{reg}\langle \frac{x^{\mu}}{x_1^{\mu_1}} \rangle$, therefore we can assume that x_1 does not divide x^{μ}.

Denote by $\lfloor * \rfloor$ the greatest lower integer function. For $1 \leq k \leq n$ and $j \geq 0$ define

$$d_{kj}(\mu) = \sum_{i=1}^{k} \lfloor \frac{\mu_i}{p^j} \rfloor.$$

For every k such that $\mu_k \neq 0$, let $s_k = \lfloor \log_p \mu_k \rfloor$. Set

$$D_k = d_{k s_k}(\mu) p^{s_k} + (k-1)(p^{s_k} - 1).$$

What is the regularity and what the projective dimension of $\langle u \rangle$?

Pardue showed that $\text{reg}\langle x^{\mu} \rangle = \mu_1 + \text{reg}\langle \frac{x^{\mu}}{x_1^{\mu_1}} \rangle$, therefore we can assume that x_1 does not divide x^{μ}.

Denote by $\lfloor * \rfloor$ the greatest lower integer function. For $1 \leq k \leq n$ and $j \geq 0$ define

$$d_{kj}(\mu) = \sum_{i=1}^{k} \lfloor \frac{\mu_i}{p^j} \rfloor.$$

For every k such that $\mu_k \neq 0$, let $s_k = \lfloor \log_p \mu_k \rfloor$. Set

$$D_k = d_{ks_k}(\mu) p^{s_k} + (k-1)(p^{s_k} - 1).$$

The following result was conjectured by Pardue in his thesis.
Theorem 3:(Aramova–H–Popescu) If x_1 does not divide x^{μ}, then

$$\text{reg}\langle x^{\mu} \rangle = \max_{k\ \mu_k \neq 0} \{D_k\}.$$

Problem: Give a formula for the projective dimension of a principal p-Borel ideal.