Generic Initial Ideals; Lecture 3

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Outline

Almost regular sequences

Generic annihilator numbers

Extremal Betti numbers

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Then

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The linear form $y \in S_1$ is called almost regular on M, if $0 :_M y$ has finite length.

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Proof: The submodule $N = 0 :_M \mathfrak{m}^\infty \subset M$ is of finite length and M/N has positive depth. It follows that any nonzero divisor $y \in S_1$ is almost regular on M. Let $\operatorname{Ass}(M/N) = \{P_1, \ldots, P_r\}$. Since $\operatorname{depth} M/N > 0$ and since K is infinite, it follows that $P_i \cap S_1$ is a proper linear subspace of S_1 for all i. Thus $U = S_1 \setminus \bigcup_{i=1}^r (S_1 \cap P_i)$ is a nonempty Zariski open subset of S_1. Any $y \in U$ is regular on M/N and hence almost regular on M. $\checkmark$

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Definition: A sequence $\mathbf{y} = y_1, \ldots, y_r$ with $y_i \in S_1$ an almost regular sequence on M, if y_i is an almost regular element on $M/(y_1, \ldots, y_{i-1})M$ for all $i = 1, \ldots, r$.

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Indeed, by using an induction argument it suffices to show that x_n is almost regular. But this is obvious since for each $P \in \mathrm{Ass}(S/I)$ we have $P = (x_1, \ldots, x_i)$ for some i. Thus the element x_n does not belong to any associated prime ideal of S/I which is different from $\mathfrak{m} = (x_1, \ldots, x_n)$.

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Let $\mathbf{y}$ be an almost regular sequence.

We denote by $A_i(\mathbf{y}; M)$ the graded module $0 :_{M/(y_1,\dots,y_i)M} y_{i+1}$ and call the numbers

$$\alpha_{ij}(\mathbf{y}; M) = \begin{cases} \dim_K A_i(\mathbf{y}; M)_j, & \text{if } i < n, \\ \beta_{0j}(M), & \text{if } i = n. \end{cases}$$

the annihilator numbers of M with respect to the sequence $\mathbf{y}$.

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Proposition 1: Let $\mathbf{y}$ a K-basis of S_1 which is almost regular on M. Then $\alpha_i(\mathbf{y}; M) = 0$ if and only if $i < \operatorname{depth} M$.

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(a) $\mathbf{y}$ is an almost regular sequence on M.

(b) $H_j(y_1, \ldots, y_i; M)$ has finite length for all $j > 0$ and all $i = 1, \ldots, r$.

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(b) $H_j(y_1, \ldots, y_i; M)$ has finite length for all $j > 0$ and all $i = 1, \ldots, r$.

(c) $H_1(y_1, \ldots, y_i; M)$ has finite length for all $i = 1, \ldots, r$.

Proof: (a) $\Rightarrow$ (b): We prove the assertion by induction on i. We have $H_j(y_1; M) = 0$ for $j > 1$ and $H_1(y_1; M) \simeq 0 :_M y_1$. This module is of finite length by assumption.

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Now let $i > 1$. Then there is the long exact sequence of graded Koszul homology

$$\begin{aligned} &\to H_{j+1}(y_1, \ldots, y_{i-1}; M) \to H_{j+1}(y_1, \ldots, y_i; M) \to H_j(y_1, \ldots, y_{i-1}; M)(- \\ &\qquad \cdots \to H_1(y_1, \ldots, y_{i-1}; M) \to H_1(y_1, \ldots, y_i; M) \to A_{i-1}(\mathbf{y}; M)(-1) \end{aligned}$$

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Since $A_{i-1}(\mathbf{y}; M)$ has finite length and since by induction

hypothesis the modules $H_j(y_1, \dots, y_{i-1}; M)$ have finite length for all $j > 0$, the exact sequence implies that also $H_j(y_1, \dots, y_i; M)$ has finite length for all $j > 0$. $\checkmark$

Generic annihilator numbers

Theorem 1: Let $I \subset S$ be a graded ideal. With each $\gamma = (g_{ij}) \in \mathrm{GL}_n(K)$ we associate the sequence $\mathbf{y} = \gamma(\mathbf{x})$ with $y_j = \sum_{i=1}^{n} g_{ij} x_i$ for $j = 1, \ldots, n$. Then there exists a nonempty Zariski open subset $U \subset \mathrm{GL}_n(K)$ such that $\gamma(\mathbf{x})$ is almost regular for all $\gamma \in U$.

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Moreover, the open set U has the property that

$$\dim_K A_{i-1}(\gamma(\mathbf{x}); S/I)_j = \dim_K A_{i-1}(x_n, x_{n-1}, \ldots, x_1; S/\mathrm{gin}_{<_{rev}}(I))_j$$

for all i and j and all $\gamma \in U$.

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We set $\alpha_{ij}(S/I) = \alpha_{ij}(\mathbf{y}; S/I)$ for all i and j , where $\mathbf{y}$ is a generic sequence on S/I. The numbers $\alpha_{ij}(S/I)$ are called the generic annihilator numbers.

The proof Theorem 1 is based on the following

Lemma 2: Let $I \subset S$ be a graded ideal. Then for all i one has

$$\mathrm{in}_{<_{rev}}(I, x_{i+1}, \ldots, x_n) = (\mathrm{in}_{<_{rev}}(I), x_{i+1}, \ldots, x_n),$$

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For the converse inclusion it suffices to show that each monomial u in $(\mathrm{in}_{<_{rev}}(I), x_{i+1}, \ldots, x_n) : x_i$ belongs to $\mathrm{in}_{<_{rev}}((I, x_{i+1}, \ldots, x_n) : x_i)$.

We may assume that no x_j with $j > i$ divides u. Then there exists a homogeneous element $f \in I$ with $ux_i = \mathrm{in}_{<_{rev}}(f)$. Because we use the reverse lexicographic order it follows that $f = cux_i + h$ with $h \in (x_i, \ldots, x_n)$ and $c \in K \setminus \{0\}$. Write $h = g_i x_i + \cdots + g_n x_n$, and set $f_1 = cu + g_i$. Then $f_1 x_i \in (I, x_{i+1}, \ldots, x_n)$ and $\mathrm{in}_{<_{rev}}(f_1) = u$. This shows that $u \in \mathrm{in}_{<_{rev}}((I, x_{i+1}, \ldots, x_n) : x_i)$.$\checkmark$

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Corollary 1: Let $I \subset S$ be a graded ideal. Then

$$\alpha_{ij}(S/I) = \alpha_{ij}(S/\operatorname{gin}_{<_{rev}}(I)).$$

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Proof: We have
$\alpha_{ij}(S/I) = \dim_K A_i(x_n, x_{n-1}, \ldots, x_1; S/\operatorname{gin}_{<_{rev}}(I))$,
and
$\alpha_{ij}(S/\operatorname{gin}_{<_{rev}}(I)) =$
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$\dim_K A_i(x_n, x_{n-1}, \ldots, x_1; S/\operatorname{gin}_{<_{rev}}(I))$. Hence the conclusion.
$\checkmark$

Extremal Betti numbers

By using the long exact sequences of Koszul homology attached to a sequence one obtains

Proposition 3: Let M be a graded S-module and let $\mathbf{y} = y_1, \ldots, y_n$ be a K -basis of S_1 which is almost regular on M. Then

$$\beta_{i,i+j}(M) \leq \sum_{k=0}^{n-i} \binom{n-k-1}{i-1} \alpha_{kj}(\mathbf{y}; M) \quad \text{for all} \quad i \geq 0 \quad \text{and all} \quad j.$$

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We say that M has maximal Betti numbers if equality holds.

This is the case if and only if $\mathfrak{m}H_j(y_1, \ldots, y_i; M) = 0$ for all $j > 0$ and all i.

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$M = S/I$ satisfies this condition, if I is a stable ideal.

Definition. Let M be a finitely generated graded S-module and let $\mathbf{y}$ be a K-basis of S_1 which is almost regular on M. Let α_{ij} be the annihilator numbers of M with respect to $\mathbf{y}$ and β_{ij} be the graded Betti numbers of M.

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(a) An annihilator number $\alpha_{ij} \neq 0$ is called extremal if $\alpha_{k\ell} = 0$ for all pairs $(k,\ell) \neq (i,j)$ with $k \leq i$ and $\ell \geq j$.

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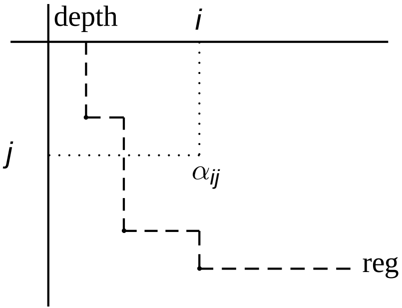
The next picture illustrates this concept.


depth
i
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α_{ij}
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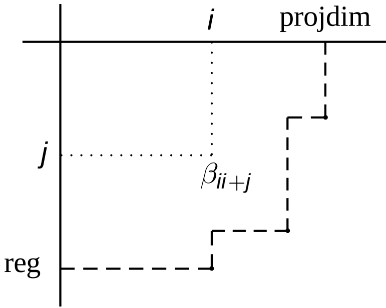
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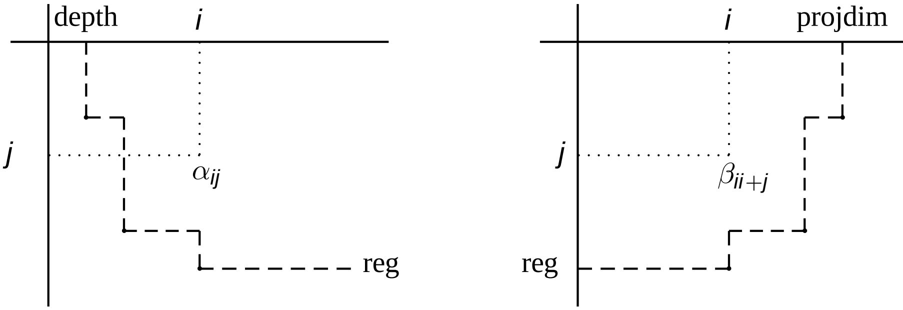

The Figure reflects the result of the next theorem

Theorem 2: (Aramova-H) Let M be a graded S -module and let $\mathbf{y}$ be a K-basis of S_1 which is almost regular on M . Let α_{ij} be the annihilator numbers of M with respect to $\mathbf{y}$ and β_{ij} be the graded Betti numbers of M.

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Then $\beta_{i,i+j}$ is an extremal Betti number of M if and only if $\alpha_{n-i,j}$ is an extremal annihilator number of M.

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Then $\beta_{i,i+j}$ is an extremal Betti number of M if and only if $\alpha_{n-i,j}$ is an extremal annihilator number of M.

Moreover, if the equivalent conditions hold, then

$$\beta_{i,i+j} = \alpha_{n-i,j}.$$

Theorem 2 together with Corollary 1 yields

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(d) $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{gin}_{<_{rev}}(I))$. In particular I has a linear resolution if and only if $\operatorname{gin}_{<_{rev}}(I)$ has a linear resolution.

Problem: Given coordinates $(i_1, j_1), \ldots, (i_r, j_r)$ with $i_1 < i_2 < \cdots < i_r$ and $j_1 > j_2 > \cdots > j_r$ and positive integers $a_1, \ldots, a_r$.

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There are some special case known by Crupi and Utano:

A lexsegment ideal can have at most 2 extremal Betti numbers.

Example: Let $I \subset K[x_1, x_2, x_3, x_4, x_5, x_6]$ be the ideal
$I = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2^3, x_2^2x_3, x_2x_3^3, x_2x_3^2x_5, x_3^5)$.

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I has two extremal Betti numbers: $\beta_{2,2+3} = 1$ and $\beta_{4,4+2}(I) = 2$.