Generic Initial Ideals; Lecture 5

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Symmetric algebraic shifting

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Note that Δ is shifted if and only if I_Δ is squarefree strongly stable.

Definition: A shifting operation on $[n]$ is a map which associates each simplicial complex Δ on $[n]$ with a simplicial complex $\mathrm{Shift}(\Delta)$ on $[n]$ and which satisfies the following conditions:

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In classical combinatorics of finite sets, Erdös, Ko and Rado introduced combinatorial shifting.

Let Δ be a simplicial complex on $[n]$. Let $1 \leq i < j \leq n$. Write $\mathrm{Shift}_{ij}(\Delta)$ for the collection of subsets of $[n]$ consisting of the sets $C_{ij}(F) \subset [n]$, where $F \in \Delta$ and where

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$$C_{ij}(F) = \begin{cases} (F \setminus \{i\}) \cup \{j\}, & \text{if } i \in F,\ j \notin F \text{ and } (F \setminus \{i\}) \cup \{j\} \notin \Delta, \\ F, & \text{otherwise.} \end{cases}$$

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Proposition 1: (a) $\mathrm{Shift}_{ij}(\Delta)$ is a simplicial complex on $[n]$, and the operation $\Delta \to \mathrm{Shift}_{ij}(\Delta)$ satisfies the conditions (S_2), (S_3) and (S_4).

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Proposition 1: (a) $\mathrm{Shift}_{ij}(\Delta)$ is a simplicial complex on $[n]$, and the operation $\Delta \to \mathrm{Shift}_{ij}(\Delta)$ satisfies the conditions (S_2), (S_3) and (S_4).

(b) There exists a finite sequence of pairs of integers $(i_1, j_1), (i_2, j_2), \ldots, (i_q, j_q)$ with each $1 \leq i_k < j_k \leq n$ such that

$$\mathrm{Shift}_{i_q j_q}(\mathrm{Shift}_{i_{q-1} j_{q-1}}(\cdots(\mathrm{Shift}_{i_1 j_1}(\Delta))\cdots))$$

is shifted.

A shifted complex which is obtained by a finite number of sequences of operations as described before will be denoted by Δ^c and is called a combinatorial shifted complex of Δ.

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Δ is not shifted because $\{1,4\} \notin \Delta$. We apply the operator $\mathrm{Shift}_{2,4}$. Then $\mathrm{Shift}_{2,4}(\Delta)$ has the facets $\{1,4\},\{2,3,4\}$. Since $\mathrm{Shift}_{2,4}(\Delta)$ is already shifted, we see that $\Delta^c = \mathrm{Shift}_{2,4}(\Delta)$.

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In general, Δ^c depends on its construction by the sequence of the operators Shift_{ij}.

Kalai's squarefree operator

Let K be a field of characteristic 0 and $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K. We work with the reverse lexicographic order $<_{rev}$ on S induced by the ordering $x_1 > \cdots > x_n$.

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Let $I \subset S$ be a squarefree monomial ideal and $\mathrm{gin}_{<_{rev}}(I)$ its generic initial ideal with respect to $<_{rev}$. Since K is of characteristic 0, it follows that $\mathrm{gin}_{<_{rev}}(I)$ is strongly stable. However, $\mathrm{gin}_{<_{rev}}(I)$ is no longer squarefree.

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Lemma 1: Let $I \subset S$ be a squarefree monomial ideal. Then

$$m(u) + \deg u \leq n + 1$$

for all monomials u belonging to $G(\mathrm{gin}_{<_{rev}}(I))$.

Proof: Since $\mathrm{gin}_{<_{rev}}(I)$ is strongly stable, the Eliahou-Kervaire formulas yield

$$\beta_{ii+j}(I) = \sum_{u \in G(\mathrm{gin}_{<_{rev}}(I))_j} \binom{m(u)-1}{i},$$

where $G(\mathrm{gin}_{<_{rev}}(I))_j$ is the set of monomials $u \in G(\mathrm{gin}_{<_{rev}}(I))$ of degree j.

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Thus in particular

$$\max\{m(u) + \deg u - 1 : u \in G(\mathrm{gin}_{<_{rev}}(I))\}$$

is the highest shift in the resolution of $\mathrm{gin}_{<_{rev}}(I)$. Since I is a squarefree monomial ideal, and since by Hochster the resolutions of squarefree ideals have only squarefee shifts it follows that the highest shift in the resolution of I is at most n.

Since the Betti number with the highest shift in the resolution on I is extremal, it follows from the theorem of Bayer-Charalambous-Popescu that the highest shift in the resolution of I and that of $\text{gin}_{<_{rev}}(I)$ coincides.

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Hence $m(u) + \deg u - 1 \leq n$ for all $u \in G(\mathrm{gin}_{<_{rev}}(I))$. $\checkmark$

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Let $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ be a monomial of S, where $i_1 \leq i_2 \leq \cdots \leq i_d$.

We set

$$u^\sigma = x_{i_1} x_{i_2+1} \cdots x_{i_j+(j-1)} \cdots x_{i_d+(d-1)}.$$

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One has

$$m(u^\sigma) - \deg u^\sigma = m(u) - 1. \tag{1}$$

The operator $u \to u^\sigma$ is called squarefree operator.

Corollary 1: Let I be a squarefree ideal of S. Then u^σ belongs to S for all $u \in G(\mathrm{gin}_{rev}(I))$.

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Let $I \subset S$ be strongly stable ideal with $G(I) = \{u_1, \ldots, u_s\}$. We write I^σ for the squarefree monomial ideal generated by the monomials $u_1^\sigma, \ldots, u_s^\sigma$.

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Lemma 2: If $I \subset S$ is strongly stable with $G(I) = \{u_1, \ldots, u_s\}$, then I^σ is squarefree strongly stable with $G(I^\sigma) = \{u_1^\sigma, \ldots, u_s^\sigma\}$.

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Next we show why I^σ is squarefree strongly stable. We take a monomial $u = x_{i_1} \cdots x_{i_d} \in G(I)$ together with $u_0 = (x_b u^\sigma)/x_{i_a+(a-1)}$, where x_b does not divide u^σ and where $b < i_a + (a-1)$ and $a \in [d]$. We claim $u_0 \in I^\sigma$.

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Choose $p < a$ such that $i_p + (p-1) < b < i_{p+1} + p$. (Here $i_0 = 1$). Let

$$v = (\prod_{j=1}^{p} x_{i_j}) x_{b-p} (\prod_{j=p+1}^{a-1} x_{i_j-1})(\prod_{j=a+1}^{d} x_{i_j}).$$

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Since $b - p < i_{p+1} \leq i_a$ and since I is strongly stable, the monomial v belongs to I. One has $v^\sigma = (x_b u^\sigma)/x_{i_a+(a-1)} = u_0$. Let, say, $v = x_{\ell_1} \cdots x_{\ell_d}$ with $\ell_1 \leq \cdots \leq \ell_d$.

Again, since I is strongly stable, it follows that $w = x_{\ell_1} \cdots x_{\ell_c} \in G(I)$ for some $c \leq d$. Since w^σ divides $v^\sigma = u_0$, one has $u_0 \in I^\sigma$, as desired. ✓

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Let Δ be a simplicial complex on $[n]$. Since the base field K is of characteristic 0, we have that $\mathrm{gin}_{<_{rev}}(I_\Delta)$ is strongly stable. Thus $(\mathrm{gin}_{<_{rev}}(I_\Delta))^\sigma$ is a squarefree strongly stable ideal of S.

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Definition: The symmetric algebraic shifted complex of Δ is defined to be the shifted complex Δ^s on $[n]$ with

$$I_{\Delta^s} = (\mathrm{gin}_{<_{rev}}(I_\Delta))^\sigma.$$

Lemma 3: If $I \subset S$ is a strongly stable ideal, then $\beta_{ii+j}(I) = \beta_{ii+j}(I^{\sigma})$ for all i and j.

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It is clear that the operation $\Delta \to \Delta^s$ satisfies condition (S_1). Lemma 3 implies that it satisfies also condition (S_3). Condition (S_4) is easy to see. That condition (S_2) is satisfied follows from

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Theorem 1: Let $I \subset S$ be a squarefree strongly stable ideal. Then

$$I = \mathrm{gin}_{<_{rev}}(I)^\sigma.$$

Proposition 2: Let I be a strongly stable monomial ideal. Then one has $\mathrm{gin}_{<_{rev}}(I^\sigma) = I$. In particular, the squarefree operator establishes a bijection between the strongly stable ideals and the squarefree strongly stable ideals.

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Proof: Let $J = \text{gin}_{<_{rev}}(I^\sigma)$. Then J is strongly stable and by Theorem 1 one has $J^\sigma = I^\sigma$. Therefore $G(J^\sigma) = G(I^\sigma)$. By Lemma 2 it follows that $G(J) = G(I)$. $\checkmark$

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(a) the ijth Betti number of I_Δ is extremal if and only if the ijth Betti number of I_{Δ^s} is extremal;

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(b) The corresponding extremal Betti numbers of I_Δ and I_{Δ^s} are equal.

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Proof: Since shifting operators preserve f-vectors, it follows that $\dim K[\Delta] = \dim K[\Delta^s]$. Now Theorem 2 implies that

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Thus $\operatorname{depth} K[\Delta] = \operatorname{depth} K[\Delta^s]$ by the Auslander–Buchbaum theorem. This shows the equivalence of statements (a) and (b).

(b) $\Leftrightarrow$ (c): We first observe that I_{Δ^s} is squarefree strongly stable. Thus we have to show that a squarefree strongly stable ideal I is Cohen–Macaulay if and only if all minimal prime ideals of I have the same height.

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Since $I^\vee$ is squarefree strongly stable, this is the case if and only if $I^\vee$ has linear resolution. By the Theorem of Eagon–Reiner this is equivalent to saying that I is a Cohen–Macaulay ideal.$\checkmark$

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Here is my question: Let I be a monomial ideal and let a be the highest exponent appearing among the generators of I. Is it true that $\operatorname{gin}_{<_{rev}}(I)$ is independent of $\operatorname{char} K$ for $\operatorname{char} K > a$?

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Really?

$\operatorname{char} K = 0$ was required to make sure that $\operatorname{gin}_{<_{rev}}(I_\Delta)$ is strongly stable.

Here is my question: Let I be a monomial ideal and let a be the highest exponent appearing among the generators of I. Is it true that $\operatorname{gin}_{<_{rev}}(I)$ is independent of $\operatorname{char} K$ for $\operatorname{char} K > a$?

If yes, then Δ^s is defined in all characteristics.

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The exterior shifted simplicial complex Δ^e is defined by the equation

$$J_{\Delta^e} = \operatorname{gin}_{<_{rev}}(J_\Delta).$$

One expects the following inequalities

$$\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\Delta^s}) \leq \beta_{ij}(I_{\Delta^e}) \leq \beta_{ij}(I_{\Delta^c}) \leq \beta_{ij}(I_{\Delta^{lex}}),$$

where Δ^{lex} is the simplicial complex whose Stanley–Reisner ideal is the unique squarefree lexsegment ideal with the same Hilbert function as I_Δ.

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In this chain of inequalities, the inequality $\beta_{ij}(I_{\Delta^s}) \leq \beta_{ij}(I_{\Delta^e})$ and even the inequality $\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\Delta^e})$ is not known.