Computational Commutative Algebra
Castelnuovo-Mumford regularity

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- is one of the most important invariants of a graded module.
- is related to the theory of syzygies which connects the qualitative study of algebraic varieties and commutative rings with the study of their defining equations.
- is related to the local cohomology theory
- is a good measure of the complexity of computing Gröbner bases.
- is a very active area of research which involves specialists working in commutative algebra, algebraic geometry and computational algebra.

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Contents

1 Castelnuovo Mumford Regularity via minimal free resolutions and Hilbert functions

Notations

- Denote
$$P = k[x_1, \dots, x_n]$$
a polynomial ring over a field k with deg $x_i = 1$

$P_j :=$ k-vector space generated by the forms of P of degree j.

- M a finitely generated graded P-module (such as an homogeneous ideal I or P/I), i.e.
$$M = \oplus_i M_i$$
as abelian groups and $P_j M_i \subseteq M_{i+j}$ for every i, j.

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$$M(d)_i := M_{i+d}.$$

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Hilbert Function

Definition

The numerical function

$$HF_M(j) := \dim_k M_j$$

is called the Hilbert function of M.

Assume $M = P/I$ where I is an homogeneous ideal of P. Then

$$HF_{P/I}(j) = \dim_k (P/I)_j$$

An important motivation arises in projective geometry: let $X \subseteq \mathbb{P}^r$ be a projective variety defined by $I = I(X) \subseteq P = k[x_0, \dots, x_r]$.

If we write $A(X) = P/I(X)$ for the homogeneous coordinate ring of X :

$$HF_X(j) = \dim_k A(X)_j = \dim_k P_j - \dim_k I_j = \binom{r+j}{r} - \dim_k I_j$$

$\dim_k I_j$ --→ the "number" of hypersurfaces of degree j vanishing on X.

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Let τ be a term ordering on $\mathbb{T}^n$, then $G = \{f_1, \ldots, f_s\}$ is a τ-Gröbner basis of I if

$$\mathrm{Lt}_\tau(I) :=< \mathrm{Lt}_\tau(f) \ : \ f \in I >= \{\mathrm{Lt}_\tau(f_1), \ldots, \mathrm{Lt}_\tau(f_s)\}$$

The residue classes of the elements of $\mathbb{T}^n \setminus \mathrm{Lt}_\tau(I)$ form a k-basis of P/I.

Proposition (Macaulay)

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Hilbert Polynomial, Hilbert Series

- $HF_M(j)$ for $j \gg 0$ agrees with $HP_M(X)$ a polynomial of degree $d-1$ where $d =$ Krull dimension of M (> 0).

  $HP_M(j)$ is called Hilbert Polynomial and it encodes several asymptotic information on M.

- A more compact information can be encoded by the Hilbert Series

$$HS_M(z) := \sum_{j \geq 0} HF_M(j) z^j = \frac{h_M(z)}{(1-z)^d} \quad (\text{Hilbert} - \text{Serre})$$

  where $h_M(1) = e > 0$ is the multiplicity of M and $d = \dim M$.

- Define

$$\text{reg-index}(M) := \max\{j : \; HF_M(j) \neq HP_M(j)\}$$

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Example

(CoCoA)

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-------------------------------
-- The current ring is R ::= Q[x,y,z];
-------------------------------
I:=Ideal(x^2,xy,xz, y^3);
H:=Hilbert(R/I);
H;
H(0) = 1
H(t) = 3   for t >= 1
-------------------------------
HilbertPoly(R/I);
3
-------------------------------
Poincare(R/I);   (or HilbertSeries(R/I);)
(1 + 2x) / (1-x)
-------------------------------
```

Minimal free resolutions

- A graded free resolution of M as a graded P-module is an exact complex ($\ker f_{j-1} = \operatorname{Im} f_j$ for every j)

$$\mathbb{F}: \quad \dots F_h \xrightarrow{f_h} F_{h-1} \xrightarrow{f_{h-1}} \dots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

where F_i are free P-modules and f_i are homogeneous homomorphisms (of degree 0).

- $\mathbb{F}$ is minimal if for every $i \geq 1$

$$\operatorname{Im} f_i \subseteq mF_{i-1}$$

where $m = (x_1, \dots, x_n)$.

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Existence of minimal graded free resolutions

- Every finitely generated P-module admits a minimal free resolution:

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- We are interested in building a graded minimal P-free resolution:

$M =< m_1, \dots, m_t >_P$ minimally generated with $\deg m_j = a_{0j}$.
Define the homogeneous epimorphism:

$$F_0 = \oplus_j P(-a_{0j}) \xrightarrow{f_0} M \to 0$$

$$e_j \quad \rightsquigarrow \quad m_j$$

By the minimality of the system of generators

$$\operatorname{Ker} f_0 \subseteq mF_0$$

We can iterate the procedure

$$0 \to \operatorname{Ker} f_i \to F_i = \oplus_i P(-a_{ij}) \xrightarrow{f_i} Kerf_{i-1} \to 0$$

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Example

$I = (x^2, xy, xz, y^3)$ in $P = k[x, y, z]$. Define

$$P(-2)^3 \oplus P(-3) \xrightarrow{f_0} I \to 0$$

$$\begin{aligned} e_1 &\rightsquigarrow x^2 \\ e_2 &\rightsquigarrow xy \\ e_3 &\rightsquigarrow xz \\ e_4 &\rightsquigarrow y^3 \end{aligned}$$

$Syz_1(I) = \operatorname{Ker} f_0$ is generated by $s_1 = ye_1 - xe_2$; $s_2 = ze_1 - xe_3$; $s_3 = ze_2 - ye_3$; $s_4 = y^2e_2 - xe_4$. Define

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$$e'_i \rightsquigarrow s_i$$

$Syz_2(I) = \operatorname{Ker} f_1$ is generated by $s = ze'_1 - ye'_2 + xe'_3$.
A minimal free resolution of I as P-module is given by:

$$0 \to P(-4) \xrightarrow{f_2} P(-3)^3 \oplus P(-4) \xrightarrow{f_1} P(-2)^3 \oplus P(-3) \xrightarrow{f_0} I \to 0.$$

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Minimal graded free resolution

A minimal graded free resolution of M as P-module can be presented as follows:

$$\mathbb{F}: \quad \cdots \oplus_{j=1}^{\beta_h} P(-a_{hj}) \xrightarrow{f_h} \oplus_{j=1}^{\beta_{h-1}} P(-a_{h-1j}) \xrightarrow{f_{h-1}} \ldots \xrightarrow{f_1} \oplus_{j=1}^{\beta_0} P(-a_{0j}) \xrightarrow{f_0} M \to 0$$

It will be useful rewrite the resolution as follows:

$$\cdots \to F_i = \oplus_{j \geq 0} P(-j)^{\beta_{ij}} \to \cdots \to \oplus_{j \geq 0} P(-j)^{\beta_{0j}} \to M$$

1) $\beta_{ij} \geq 0$
2) $\beta_{ij} =$ cardinality of the shift $(-j)$ in position i $(\beta_i = \sum \beta_{ij})$

Question. Does β_{ij} (hence a_{ij}) depend on the maps f_i of the resolution?

We remind that in proving the existence of a minimal free resolution we can choose different system of generators of the kernels, hence different maps.

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$$\cdots \to F_i = \oplus_{j \geq 0} P(-j)^{\beta_{ij}} \to \cdots \to \oplus_{j \geq 0} P(-j)^{\beta_{0j}} \to M$$

1) $\beta_{ij} \geq 0$
2) $\beta_{ij} =$ cardinality of the shift $(-j)$ in position i $(\beta_i = \sum \beta_{ij})$

Question. Does β_{ij} (hence a_{ij}) depend on the maps f_i of the resolution?

We remind that in proving the existence of a minimal free resolution we can choose different system of generators of the kernels, hence different maps.

Minimal graded free resolution

A minimal graded free resolution of M as P-module can be presented as follows:

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Basic facts

We prove that the graded Betti numbers are uniquely determined by M.

Proposition

$$\beta_{ij} = \beta_{ij}(M) = dim_k Tor_i^P(M,k)_j$$

and we call these integers graded Betti numbers of M.

In fact

$$Tor_i^P(M,k) = H_i(\mathbb{F} \otimes P/m)$$

By the minimality of $\mathbb{F}$ the maps of the new complex $\mathbb{F} \otimes P/m$ are trivial, hence we have

$$Tor_i^P(M,k)_j = [\oplus_{m \geq 0} P(-m)^{\beta_{im}} \otimes P/m]_j = [\oplus_{m \geq 0} k(-m)^{\beta_{im}}]_j =$$

$$= \oplus_{m \geq 0} (k_{j-m})^{\beta_{im}} \underset{m=j}{=} k^{\beta_{ij}}$$

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The resolution fixes the Hilbert Function

Let I be an homogeneous ideal of P.

Proposition

If $\beta_{ij} = \beta_{ij}(P/I)$ are the graded Betti numbers of P/I, then the Hilbert series of P/I is given by

$$HS_{P/I}(z) = \frac{1 + \sum_{ij}(-1)^{i+1}\beta_{ij}z^j}{(1-z)^n}$$

If we consider the previous example $I = (x^2, xy, xz, y^3)$ in $P = k[x, y, z]$. We have seen that a minimal free resolution of I as P-module is given by:

$$0 \to P(-4) \to P(-3)^3 \oplus P(-4) \to P(-2)^3 \oplus P(-3) \to P \to P/I \to 0.$$

Since $HS_{P(-d)^\beta}(z) = \frac{\beta z^d}{(1-z)^n}$, then

$$HS_{P/I}(z) = \frac{1 - 3z^2 - z^3 + 3z^3 + z^4 - z^4}{(1-z)^3} = \frac{1+2z}{1-z}$$

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Exercise

Consider

$$X = \{P_1, \ldots, P_4\} \subseteq \mathbb{P}^2$$

four distinct points in the plane.

Denote $A(X) = k[x_0, x_1, x_2]/I(X)$ the corresponding coordinate ring. Prove:

- the Hilbert polynomial of a set of four points, no matter what the configuration, is a constant polynomial $HP_X(n) = 4$.
- the Hilbert function of X depends only on whether all four points lie on a line.
- The graded Betti numbers of the minimal resolution, in contrast, capture all the remaining geometry: they tell us whether any three of the points are collinear as well.

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Hilbert's Syzygy Theorem

Theorem (Hilbert's Syzygy Theorem)

Every finitely generated P-module has a ***finite*** *free resolution (of length $\leq n$)*

We remind that $Tor_i(k, M) = H_i(\mathbb{K} \otimes M)$ where $\mathbb{K}$ is a minimal free resolution of $k = P/(x_1, \ldots, x_n)$ as P-module.

Hence we consider the Koszul complex of $(x_1, \ldots, x_n)$.:

$$\mathbb{K} : 0 \to P(-n)^{\binom{n}{n}} \to P(-n+1)^{\binom{n}{n-1}} \to \cdots \to P(-1)^{\binom{n}{1}} \to P$$

We deduce

$$Tor_i(k, M) = H_i(\mathbb{K} \otimes M) = 0$$

for every $i \geq n+1$ ($K_i = 0$ for $i \geq n+1$).

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Auslander-Buchsbaum formula

If M has the following minimal P-free resolution:

$$0 \to F_h = \oplus_{j\geq 0} P(-j)^{\beta_{hj}} \to \cdots \to \oplus_{j\geq 0} P(-j)^{\beta_{0j}} \to M$$

Define the Projective dimension (or Homological dimension) of M

$$pd(M) := \max\{i : \beta_{ij}(M) \neq 0 \text{ for some } j\}$$

that is $h =$ length of the resolution.

Theorem (Auslander-Buchsbaum)

$$pd_P(M) = n - \operatorname{depth}(M)$$

where $\operatorname{depth}(M) =$ *length of a (indeed any) maximal M-regular sequence in* $m = (x_1, \dots, x_n)$.

M is Cohen-Macaulay $\iff \operatorname{depth} M = \dim M \iff \operatorname{pd}_P(M) = n - \dim M$.

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Betti Diagram

The numerical invariants in a minimal free resolution can presented by using "a piece of notation" introduced by Bayer and Stillman: the Betti diagram.

This is a table displaying the numbers β_{ij} in the pattern

	0	1	2	$\dots$	i
0 :	β_{00}	β_{11}	β_{22}	$\cdots$	β_{ii}
1 :	β_{01}	β_{12}	β_{23}	$\cdots$	β_{ii+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s :	β_{0s}	β_{1s+1}	β_{2s+2}	$\cdots$	β_{ii+s}
$\sum$	β_0	β_1	β_2	$\cdots$	β_i

with β_{ij} in the i-th column and $(j-i)$-th row.

Thus the i-th column corresponds to the i-th free module

$$F_i = \oplus_j P(-j)^{\beta_{ij}}.$$

Example

(CoCoA)

```
  Use R ::= QQ[t,x,y,z];
  I := Ideal(x^2-yt,xy-zt,xy);
  Res(I);
0 --> R^2(-5) --> R^4(-4) --> R^3(-2)
-------------------------------
  BettiDiagram(I);
         0    1    2
 --------------------
  2:     3    -    -
  3:     -    4    2
 --------------------
 Tot:    3    4    2
 -------------------------------
```

Definition

Given a minimal P-free resolution of M :

$$\mathbb{F} : \ldots.. \qquad \to F_i = \oplus P(-j)^{\beta_{ij}(M)} \to \cdots \to F_0 = \oplus P(-j)^{\beta_{0j}(M)}$$

the Castelnuovo-Mumford regularity of M

$$reg(M) = \max_i \{ j - i : \beta_{ij}(M) \neq 0 \}$$

Equivalently if we write

$$\mathbb{F} : \quad \cdots \oplus_{j=1}^{\beta_h} P(-a_{hj}) \xrightarrow{f_h} \oplus_{j=1}^{\beta_{h-1}} P(-a_{h-1j}) \xrightarrow{f_{h-1}} \ldots \xrightarrow{f_1} \oplus_{j=1}^{\beta_0} P(-a_{0j}) \xrightarrow{f_0} M \to 0$$

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If we consider THE example

$$I = (x^2, xy, xz, y^3) \subseteq P = k[x, y, z].$$

We have seen that a minimal free resolution of I as P-module is given by:

$$0 \to F_2 = P(-4) \xrightarrow{f_2} F_1 = P(-3)^3 \oplus P(-4) \xrightarrow{f_1} F_0 = P(-2)^3 \oplus P(-3) \xrightarrow{f_0} I \to 0.$$

Then

- $pd(I) = 2$
- $reg(I) = 3 =$ max degree of a minimal generator.
- $\dim P/I = 1$ (we know that $HS_{P/I}(z) = \frac{1+2z}{1-z}$).

  Hence P/I is not Cohen-Macaulay since $pd(P/I) = 3 > 3 - \dim P/I = 2$.
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```
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 4
 -------------------------------
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-------------------------------
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-------------------------
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 3:     3    5    -    -
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-------------------------
Tot:    5    8    6    2
-------------------------------
```

Remarks

- (Exercise) If M has finite length, then $reg(M) = \max\{j \ : \ M_j \neq 0\}$.
- $\mathrm{reg}(I) = \mathrm{reg}(P/I) + 1 \geq$ maximum degree of a minimal generator of I
- $\mathrm{reg}(P/I)$ **coincides with the last non-zero row in the Betti diagram**

Stillman's conjecture

$R = k[x_1, ..., x_n]/I = (f_1, \dots, f_r)$ where f_i are forms of degree d_i.

In general $\mathrm{pd}(I)$, as well $\mathrm{reg}(I)$, can grew relatively fast as one increases the number of generators and the degrees.

Conjecture (Stillman)

There is an upper bound, independent of n, on $pd(I)$, for any ideal I generated by r homogeneous polynomials of given degrees.

Ananyan-Hochster (2011): Positive answer if $d_i \leq 2$.

Equivalently to:

Conjecture (Caviglia-Kumini)

There is an upper bound, independent of n, on the Castelnuovo-Mumford regularity for any ideal I generated by r homogeneous polynomials of given degrees.

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Exercises

Exercise Consider the homogeneous coordinate ring of the “twisted cubic”:

$$R = K[s^3, s^2t, st^2, t^3]$$

- Prove that $R = P/I$ where $P = K[x_0, \dots, x_3]$ and $I = I_2\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$
- Prove that R is CM
- Compute $\mathrm{HF}_R(j)$, $\operatorname{reg}(R)$
- Compare $\operatorname{reg}(I)$ and $\operatorname{reg}(Lt_\tau(I))$ with τ any term ordering

Exercise Consider the homogeneous coordinate ring of the smooth rational quartic in $\mathbb{P}^3$

$$R = K[s^4, s^3t, st^3, t^4]$$

- Prove that $R \simeq P/I$ where $P = K[x_0, \dots, x_3]$ and $I = I_2\begin{pmatrix} x_0 & x_1^2 & x_1x_3 & x_2 \\ x_1 & x_0x_2 & x_2^2 & x_3 \end{pmatrix}$
- Prove that R is not CM
- Compute $\operatorname{reg}(I)$