Computational Commutative Algebra
Castelnuovo-Mumford regularity

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Tehran, 2-7 July 2011

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- We say that M is m-regular for some integer m if

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$$M \text{ is } m\text{-regular} \iff \beta_{ij}(M) = 0 \quad \forall j \geq i + m + 1$$

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Alternative definitions

Let $\mathbb{F} = \{F_i\}$ be a graded minimal free resolution of M.

$$M \text{ is } m\text{-regular} \implies F_i \text{ has no generators in degrees} \geq m + i + 1$$

Consider $Hom(\mathbb{F}, P)$ and denote $F_i^* = Hom_P(F_i, P)$, then

$$M \text{ is } m\text{-regular} \implies [F_i^*]_{\leq -m-i-1} = 0$$

$$\downarrow$$

$$Ext_P^i(M, P) = H_i(Hom(\mathbb{F}, P))$$

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$$Ext^i_P(M, P) = H_i(Hom(\mathbb{F}, P))$$

Regularity in terms of Ext's

$$reg(M) := \min\{m : \ Ext^i_P(M,P)_j = 0 \ : \forall j \le -m-i-1\}$$

The above equality is hard to apply because in principle infinitely many conditions must be checked. We introduce a new definition given by Mumford for sheaves:

M is weakly m-regular if for every i

$$Ext^i_P(M,P)_{-m-i-1} = 0$$

If either $depthM > 0$ or $M = P/I$ then

$$reg(P/I) := \min\{m : \ Ext^i_P(P/I,P)_{-m-i-1} = 0 \ \}$$

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In terms of the Local Cohomology

Denote by $H^i_m(M)$ the local cohomology module with support in m, $0 \leq i \leq d = \dim M$.

By using the local duality (Eisenbud, A 4.2)

$$H^i_m(M)_j \simeq Ext_P^{n-i}(M, P)_{-j-n}$$

We recall that $H^i_m(M)$ are Artinian and we let

$$end(H^i_m(M)) := \max\{j : \; H^i_m(M)_j \neq 0\}$$

$(\max 0 = -\infty)$

$$reg(M) = \max\{end(H^i_m(M)) + i \; : \; 0 \leq i \leq d\}$$

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By Grothendieck-Serre's formula (Bruns-Herzog Theor. 4.4.3)

$$HP_M(i) - HF_M(i) = \sum_{j=0}^{d} (-1)^{j+1} \lambda(H^j_m(M)_i)$$

As a consequence

$$HP_M(i) = HF_M(i) \quad \forall i > reg(M)$$

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Regularity and exact sequences

This approach gives a quite easy proof of the following

Proposition

Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of graded finitely generated P-modules (homogeneous maps), then

1) $reg(A) \leq \max(reg(B), reg(C) + 1)$
2) $reg(B) \leq \max(reg(A), reg(C))$
3) $reg(C) \leq \max(reg(A) - 1, reg(B))$
4) *If A has finite length, then* $reg(B) = \max(reg(A), reg(C))$.

Hint: consider the long exact sequence

$$\cdots \to Ext^{j-1}(A, P) \to Ext^{j}(C, P) \to Ext^{j}(B, P) \to$$

$$\to Ext^{j}(A, P) \to Ext^{j+1}(C, P) \to \dots$$

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Regularity and linear resolutions

Definition

I has a d-linear resolution if it is generated in one degree, say d, and $\beta_{ij}(I) = 0$ for all $j \neq i + d$. If this is the case

$$reg(I) = d.$$

$$0 \to P^{\beta_h}(-d-h) \to \cdots \to P^{\beta_1}(-d-1) \to P^{\beta_0}(-d) \to I \to 0$$

The matrices associated to the maps of the resolution have linear entries (or zero).

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Regularity and linear resolution

Proposition

Set $I_{\geq j} := I \cap m^j$.

$$r = reg(I) \implies I_{\geq j} \text{ has j-linear resolution } \forall j \geq r$$

Important steps: • $I_{<r>}$ has r-linear resolution

• M has d-linear resolution $\implies$ mM has $(d+1)$-linear resolution.

It is enough to consider the exact sequence of graded modules

$$0 \to mM \to M \to M/mM \to 0$$

Then by the exact sequence

$$reg(mM) \leq \max\{reg(M), reg(M/mM)+1\} = \max\{d, d+1\}$$

On the other hand $reg(mM) \geq d+1 = indeg\,(mM)$.

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```
Use P ::= Q[x,y,z];
I := Ideal(x^2,xy,xz,y^3);
Reg(I);
3
-------------------------------
Res(I);
0 --> P(-4) --> P^3(-3)(+)P(-4) --> P^3(-2)(+)P(-3)
-------------------------------

J:=Intersection(I,Ideal(x,y,z)^3);

Res(J);
-------------------------------
0 --> P^3(-5) --> P^9(-4) --> P^7(-3)
```

Regularity and hyperplane sections

Let $F \in P$ be homogeneous such that $0 :_M F$ has finite length, by using the comparison between regularities in exact sequences, we get

$$reg(M) = \max(reg(0 :_M F), reg(M/FM) - \deg F + 1)$$

(Actually it is enough $\dim(0 :_M F) \leq 1$)

- If $L \in P_1$ is M-regular, then

$$reg(M) = reg(M/LM)$$

- If L is a linear *filter regular element* $(M_n \xrightarrow{\cdot L} M_{n+1}$ injective $n \gg 0)$

$$reg(M) = \max\{reg(0 : L), reg(M/LM)\} \geq reg(M/LM)$$

(e.g. $\dim M > 0$, $|K| = \infty$ and L a generic linear form)

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Regularity of a CM module

Proposition

Let M be a Cohen-Macaulay graded finitely generated P-modules of dimension d

1) $reg(M) = \deg(h_M(z))$ *where $h_M(z)$ is the h-polynomial of M* *($HS_M(z) = \frac{h_M(z)}{(1-z)^d}$)*
2) $reg(M) = reg\text{-}index(M) + d$

Proof: ($|k| = \infty$) Let $J = (L_1, \ldots, L_d) \subseteq P$ the ideal generated by a maximal M-regular sequence of linear forms. We know that

$$reg(M) = reg(M/JM)$$

Now M/JM is an Artinian module and

$$reg(M/JM) = \max\{n : (M/JM)_n \neq 0\} = \deg(HS_{M/JM}(z)) = \deg(h_M(z))$$

since $HS_M(z) = \frac{HS_{M/JM}(z)}{(1-z)^d}$. Hence $reg(M) = reg\text{-}index(M/JM) = reg\text{-}index(M) + d$.

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Regularity and sums, product, intersection of ideals

Let I, J homogeneous ideals, there are the following exact sequences:

$$0 \to P/I \cap J \to P/I \oplus P/J \to P/I + J \to 0$$

$$0 \to I \cap J/IJ \to P/IJ \to P/I \cap J \to 0$$

We can prove

Theorem

If $(I \cap J)/IJ$ is a module of dimension at most 1, then

1) $reg(I + J) \leq reg(I) + reg(J) - 1$
2) $reg(I \cap J) \leq reg(I) + reg(J)$
3) $reg(IJ) \leq reg(I) + reg(J)$.

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Regularity and sums, product, intersection of ideals

Let I, J homogeneous ideals, there are the following exact sequences:

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- G. Caviglia gave an example with $\dim(I \cap J)/IJ = 2$ and $reg(I + J) \geq reg(I) + reg(J)$

The possibility of extending 2) and 3) to any number of ideals is still unclear.

- Conca and Herzog: If $I_1, \dots, I_r$ are generated by linear forms, then

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I. Swanson: There exists D such that for every $q \geq 1$

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The problem of bounding $reg(I^q)$ is also related to the regularity of

$$\mathcal{R}(I) = \oplus_q I^q$$

This problem seemed to be hard. So it came as a surprise the following result

Theorem (Cutkosky, Herzog, Trung; Hoa, Herzog, Trung)

Let $d(I)$ denote the maximum degree of I

- $\exists\, e \in \mathbf{N} \;:\; reg(I^q) \leq q\, d(I) + e$ *for every* $q \geq 1$.
- $\exists\, e \in \mathbf{N}$ *and* $c \leq d(I) \;:\; reg(I^q) = c\, q + e$ *for every* $q \gg 0$.

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Exercises

For monomial ideals, there are some more results in terms of better understood invariants of I.

Exercise 1. Let $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ with $m \leq n$ and

$$I = (x_1^{a_1}, \ldots, x_m^{a_m}).$$

Then

$$reg(I) = a_1 + \cdots + a_m - m + 1.$$

Exercise 2. Under the above assumptions:

- $reg(I^q) = qa_1 + a_2 + \cdots + a_m - m + 1$
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Regularity of the radical

Ravi proved that if I is a monomial ideal, then

$$reg(\sqrt{I}) \leq reg(I)$$

Problem. Find different classes of ideals for which $reg(\sqrt{I}) \leq reg(I)$.

Chardin-D'Cruz produced examples where $reg(\sqrt{I})$ is the cube of $reg(I)$.

Problem.(Peeva-Stillman) Is $reg(\sqrt{I})$ bounded by a (possibly polynomial) function of $reg(I)$?

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Example

Example. [Chardin-D'Cruz] Let n, m be positive integers and let

$$I_{m,n} = (x^m t - y^m z, z^{n+2} - xt^{n+1}) \subseteq K[x, y, z, t]$$

The following equalities hold

1. $\operatorname{reg}(I_{m,n}) = m + n + 2$ (complete intersection)
2. $\operatorname{reg}(\sqrt{I_{m,n}}) = m \cdot n + 2$

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$P = k[x_1, \dots, x_n]$, $|k| = \infty$, τ a monomial order

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- $\beta_{ij}(I) \leq \beta_{ij}(Lt_\tau(I)) \leq \beta_{ij}(Lex(I))$ (Bigatti, Hulett, Pardue)
- $\beta_{ij}(I) \leq \beta_{ij}(gin_{revlex}(I)) \leq \beta_{ij}(gin_\tau(I))$ (Conca)

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Properties of $\tau =$ RevLex

Let I be an homogeneous ideal in $P = k[x_1, \dots, x_n]$, $\tau =$ RevLex

- $F \in P$ homogeneous

$$Lt_\tau(F) \in (x_s, \dots, x_n),\ 1 \le s \le n \implies F \in (x_s, \dots, x_n)$$

- $Lt_\tau(I + (x_n)) = Lt_\tau(I) + (x_n)$
- $Lt_\tau(I : x_n) = Lt_\tau(I) : x_n$
- $x_n, \dots, x_s$ is a P/I-regular sequence $\iff$ $x_n, \dots, x_s$ is a $P/Lt_\tau(I)$-regular sequence

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Properties of Borel type ideals

Let I be a Borel type ideal in $P = k[x_1, \ldots, x_n]$:

- For any $j = 1, \ldots, r$

$$I : x_j^\infty = I : (x_1, \ldots, x_j)^\infty$$

  (weakly stable, nested)

  or equivalently

- If $\mathcal{P}$ is an associated prime of I, then $\mathcal{P} = (x_1, \ldots, x_j)$ for some j.

Hence if I is of Borel type and $\dim P/I > 0$, then $I : x_n/I$ is of finite length.

Gin(I): the generic initial ideal

For a generic $g \in GL_n(K)$, $Lt_\tau(g(I))$ is *constant:*

Theorem (Galligo, Bayer-Stilmann)

There exists $U \neq \emptyset$ a Zariski-open subset of $GL_n(k)$ such that

$$Lt_\tau(g(I)) = Lt_\tau(h(I))$$

for every $g, h \in U$.

Set

$$gin_\tau(I) := Lt_\tau(g(I)) \quad for \ \ every \ \ g \in U$$

$gin_\tau(I)$ *is a Borel fixed ideal, in particular of Borel type !!!*

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for every $g, h \in U$.

Set

$$gin_\tau(I) := Lt_\tau(g(I)) \quad \text{for every} \quad g \in U$$

$gin_\tau(I)$ is a Borel fixed ideal, in particular of Borel type !!!

Gin(I): the generic initial ideal

For a generic $g \in GL_n(K)$, $Lt_\tau(g(I))$ is *constant:*

Theorem (Galligo, Bayer-Stilmann)

There exists $U \neq \emptyset$ *a Zariski-open subset of* $GL_n(k)$ *such that*

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Theorem (Bayer-Stilman)

Let $I \subseteq P$ be an homogeneous ideal, $|k| = \infty$, $\tau =$ revlex.

$$reg(P/I) = reg(P/gin_\tau(I))$$

We give here an easy proof. First underline the crucial points:

- $gin_\tau(I) = Lt_\tau(g(I))$ with $\tau =$ revlex

  Assume $d > 0$, by using the properties of $\tau =$ revlex (!!!):

- $gin_\tau(I : x_n) = gin_\tau(I) : x_n \qquad gin_\tau(I + (x_n)) = gin_\tau(I) + (x_n)$.
- since gin_τ is of Borel type $gin_\tau(I) : x_n / gin_\tau(I)$ has finite length (if an associated prime contains x_n, it is the maximal ideal).
- $\dim P/I + (x_n) = \dim P/gin_\tau(I + (x_n)) = \dim P/gin_\tau(I) + (x_n) = d - 1$.

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- The result is clear if $d = dim P/I = 0$ because they have the same HF.
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We claim:

$$reg(I : x_n/I) = reg(gin_\tau(I) : x_n/gin_\tau(I))$$

In fact $gin_\tau(I : x_n) = gin_\tau(I) : x_n$ and we deduce that they have the same Hilbert function and of finite length.

Then

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A generalization of Bayer-Stilman's Theorem

Theorem

Let $I \subseteq P$ be an homogeneous ideal, $|k| = \infty$, $\tau =$ revlex. Assume that $Lt_\tau(I)$ is of Borel type, then

$$reg(P/I) = reg(P/Lt_\tau(I))$$

Bayer-Charalambous-Popescu proved a refinement of Bayer-Stilman's Theorem (extremal Betti numbers)($\rightarrow$ Juergen's lessons).
From the above extension of Bayer-Stilman's Theorem, it is thinkable that in other situations initial ideals of Borel type could replace gin. This would be appreciated from the computational point of view.

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Bayer-Stilman's Theorem in char $k = 0$

Theorem (Bayer-Stilmann; Eliahou-Kervaire)

Let $I \subseteq P$ be an homogeneous ideal, char $k = 0$ $\tau = revlex$.

$$reg(I) = reg(gin_\tau(I)) = \text{ max degree of a generator of } gin_\tau(I)$$

It can be deduced from the following facts :

- char $k = 0 \implies gin_\tau(I)$ is a strongly stable monomial ideal

(i.e. for any monomial m, $x_i m \in J \implies x_j m \in J, \forall j \leq i$)

- By Eliahou-Kervaire's resolution of stable ideals J

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Exercise

Exercise. Consider the homogeneous ideal:
$I = (x^2 - yz + 3tu, xyz^2 + z^4, xyt - 3u^3, x^2t^2 + 4y^2u^2) \subseteq P = k[x, y, z, t, u]$.

1) Compute the regularity of I using BettiDiagram
2) Compare regularity and Betti numbers of I with those of $Lt_{revlex}(I)$. Is $Lt_{revlex}(I)$ of Borel type?
3) Compute the regularity of I using $gin(I)$.

A different approach by using a Trung's result

Definition

An element $x \in P_1$ is filter regular for P/I if

$$(P/I)_i \stackrel{\cdot x}{\to} (P/I)_{i+1}$$

is injective for $i \gg 0$.
Equivalently $x \notin \wp \ \forall \wp \in Ass(I), \ \wp \neq m$.

Hence x is filter regular iff

$$(I : x)_i = I_i \quad \forall i \gg 0 \ \text{ or } \ \text{equivalently} \quad \lambda(I : x/I) < \infty$$

For example x_n is a filter regular element for an ideal of Borel type.

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Basic idea

Remark. If x is filter regular for P/I, then

$$reg(P/I) = \max\{reg(I : x/I), reg(P/I + (x))\}$$

Definition

$y_1, \ldots, y_t$ is a filter regular sequence for P/I if y_1 is filter regular and y_i is filter regular in $P/(y_1, \ldots, y_{i-1})$ for every $i = 2, \ldots, t$.

- Let $y_1, \ldots, y_t$ be a filter regular sequence for P/I. Then $y_1, \ldots, y_t$ is a s.o.p. in P/I
- If $|k| = \infty$ then there exists a maximal filter regular sequence $y_1, \ldots, y_d$ where $d = \dim P/I$.
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Trung's result

Assume $d \geq 1$. Let $\underline{y} := y_1, \ldots, y_d$ be a sequence of linear forms. Define

$$I_i := I_{i-1} + (y_i) \quad (I_0 = I)$$

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If $\lambda(a^i_{\underline{y}}(I)) < \infty$, then

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$\underline{y} := y_1, \ldots, y_d$ is a filter-regular sequence for P/I if and only if $\lambda(a^i_{\underline{y}}) < \infty \; \forall i$.

We control the regularity in terms of these integers:

Theorem (Trung)

Let $\underline{y} := y_1, \ldots, y_d$ be a maximal filter regular sequence for P/I. Then

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$$a^i_{\underline{y}}(I) := I_{i-1} : y_i / I_{i-1}$$

If $\lambda(a^i_{\underline{y}}(I)) < \infty$, then

$$reg(a^i_{\underline{y}}(I)) = \sup\{r : \ [I_{i-1} : y_i]_r \neq [I_{i-1}]_r\}$$

with $reg(a^i_{\underline{y}}) := -\infty \quad$ if $\quad I_{i-1} : y_i = I_{i-1}$.

$\underline{y} := y_1, \ldots, y_d$ is a filter-regular sequence for P/I if and only if $\lambda(a^i_{\underline{y}}) < \infty \; \forall i$.

We control the regularity in terms of these integers:

Theorem (Trung)

Let $\underline{y} := y_1, \ldots, y_d$ be a maximal filter regular sequence for P/I. Then

$$reg(P/I) = \max\{reg(a^i_{\underline{y}}(I)) : \ 1 \leq i \leq d; reg(P/I_d)\}$$

A further generalization

Let $\underline{x} := x_n, \ldots, x_{n-d+1}$, by the properties of τ=revlex we have

$$reg(a^i_{\underline{x}}(I)) = reg(a^i_{\underline{x}}(Lt_\tau(I)))$$

Theorem

Let $\underline{x} := x_n, \ldots, x_{n-d+1}$. If $\lambda(a^i_{\underline{x}}(Lt_\tau(I))) < \infty \ \forall i$, then

$$reg(P/I) = reg(P/Lt_\tau(I))$$

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Bermejo-Gimenez, Trung's algorithm

What is the needed genericity?

- Consider a (*sparse*) change of coordinates
- Compute $a^i_{\underline{x}}(Lt_{revlex}(I))$ where $\underline{x} := x_n, \ldots, x_{n-d+1}$
- The generality is enough if $\underline{x} := x_n, \ldots, x_{n-d+1}$ is a filter regular sequence for $P/Lt_{revlex}(I))$ (equivalently for P/I), that is $a^i_{\underline{x}}(Lt_{revlex}(I)) < \infty$
- If $a^i_{\underline{x}}(Lt_{revlex}(I)) < \infty$ then

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