Computational Commutative Algebra
Castelnuovo-Mumford regularity

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1 Castelnuovo Mumford regularity: computational aspects

$P = k[x_1, \ldots, x_n]$, $|k| = \infty$, τ a monomial order

$$reg(I) \quad \longleftrightarrow \quad reg(Lt_\tau(I))$$

- I and $Lt_\tau(I)$ have the same Hilbert function
- $\beta_{ij}(I) \leq \beta_{ij}(Lt_\tau(I)) \leq \beta_{ij}(Lex(I))$ (Bigatti, Hulett, Pardue)
- $\beta_{ij}(I) \leq \beta_{ij}(gin_{revlex}(I)) \leq \beta_{ij}(gin_\tau(I))$ (Conca)

  As a consequence

- $reg(I) \leq reg(Lt_\tau(I))$

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Let I be an homogeneous ideal in $P = k[x_1, \ldots, x_n]$, $\tau = \text{RevLex}$

- $F \in P$ homogeneous

$$Lt_\tau(F) \in (x_s, \ldots, x_n),\ 1 \le s \le n \implies F \in (x_s, \ldots, x_n)$$

- $Lt_\tau(I + (x_n)) = Lt_\tau(I) + (x_n)$
- $Lt_\tau(I : x_n) = Lt_\tau(I) : x_n$
- $x_n, \ldots, x_s$ is a P/I-regular sequence $\iff$ $x_n, \ldots, x_s$ is a $P/Lt_\tau(I)$-regular sequence

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Properties of Borel type ideals

Let I be a Borel type ideal in $P = k[x_1, \ldots, x_n]$:

- For any $j = 1, \ldots, r$

$$I : x_j^{\infty} = I : (x_1, \ldots, x_j)^{\infty}$$

  (weakly stable, nested)

  or equivalently

- If $\mathcal{P}$ is an associated prime of I, then $\mathcal{P} = (x_1, \ldots, x_j)$ for some j.

Hence if I is of Borel type and $\dim P/I > 0$, then $I : x_n/I$ is of finite length.

Gin(I): the generic initial ideal

For a generic $g \in GL_n(K)$, $Lt_\tau(g(I))$ is *constant:*

Theorem (Galligo, Bayer-Stilmann)

There exists $U \neq \emptyset$ a Zariski-open subset of $GL_n(k)$ such that

$$Lt_\tau(g(I)) = Lt_\tau(h(I))$$

for every $g, h \in U$.

Set

$$gin_\tau(I) := Lt_\tau(g(I)) \quad \text{for every } g \in U$$

$gin_\tau(I)$ *is a Borel fixed ideal, in particular of Borel type !!!*

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Bayer-Stilman's Theorem

Theorem (Bayer-Stilman)

Let $I \subseteq P$ be an homogeneous ideal, $|k| = \infty$, $\tau =$revlex.

$$reg(P/I) = reg(P/gin_\tau(I))$$

We give here an easy proof. First underline the crucial points:

- $gin_\tau(I) = Lt_\tau(g(I))$ with $\tau =$revlex

  Assume $d > 0$, by using the properties of $\tau =$revlex (!!!):

- $gin_\tau(I : x_n) = gin_\tau(I) : x_n \qquad gin_\tau(I + (x_n)) = gin_\tau(I) + (x_n)$.
- since gin_τ is of Borel type $gin_\tau(I) : x_n / gin_\tau(I)$ has finite length (if an associated prime contains x_n, it is the maximal ideal).
- $\dim P/I + (x_n) = \dim P/gin_\tau(I + (x_n)) = \dim P/gin_\tau(I) + (x_n) = d - 1$.

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Proof:

- The result is clear if $d = dim P/I = 0$ because they have the same HF.
- Assume $d > 0$, by induction on d :

$$\mathrm{reg}(P/I + (x_n)) = \mathrm{reg}(P/gin_\tau(I + (x_n))).$$

Hence $reg(P/I + (x_n)) = reg(P/gin_\tau(I + (x_n))) = reg(P/gin_\tau(I) + (x_n)).$

We claim:

$$reg(I : x_n/I) = reg(gin_\tau(I) : x_n/gin_\tau(I))$$

In fact $gin_\tau(I : x_n) = gin_\tau(I) : x_n$ and we deduce that they have the same Hilbert function and of finite length.

Then

$$reg(P/I) = \max\{reg(I : x_n/I), reg(P/I + (x_n))\} =$$

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A generalization of Bayer-Stilman's Theorem

Theorem

Let $I \subseteq P$ be an homogeneous ideal, $|k| = \infty$, $\tau =$ revlex. Assume that $Lt_\tau(I)$ is of Borel type, then

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Bayer-Charalambous-Popescu proved a refinement of Bayer-Stilman's Theorem (extremal Betti numbers)($\rightarrow$ Juergen's lessons).

From the above extension of Bayer-Stilman's Theorem, it is thinkable that in other situations initial ideals of Borel type could replace gin. This would be appreciated from the computational point of view.

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Bayer-Stilman's Theorem in char $k = 0$

Theorem (Bayer-Stilmann; Eliahou-Kervaire)

Let $I \subseteq P$ be an homogeneous ideal, char $k = 0$ $\tau =$ revlex.

$$reg(I) = reg(gin_\tau(I)) = \text{ max degree of a generator of } gin_\tau(I)$$

It can be deduced from the following facts :

- char $k = 0 \implies gin_\tau(I)$ is a strongly stable monomial ideal

(i.e. for any monomial m, $x_i m \in J \implies x_j m \in J, \forall j \leq i$)

- By Eliahou-Kervaire's resolution of stable ideals J

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Bayer-Stilman's Theorem in char $k = 0$

Theorem (Bayer-Stilmann; Eliahou-Kervaire)

Let $I \subseteq P$ be an homogeneous ideal, char $k = 0$ $\tau =$ revlex.

$$reg(I) = reg(gin_\tau(I)) = \text{ max degree of a generator of } gin_\tau(I)$$

It can be deduced from the following facts :

- char $k = 0 \quad \Longrightarrow \quad gin_\tau(I)$ is a strongly stable monomial ideal

(i.e. for any monomial m, $x_i m \in J \implies x_j m \in J, \forall j \leq i$)

- By Eliahou-Kervaire's resolution of stable ideals J

Bayer-Charalambous-Popescu proved a refinement of Bayer-Stilman's Theorem (extremal Betti numbers)($\rightarrow$ Juergen's lessons)

Exercise

Exercise. Consider the homogeneous ideal:
$I = (x^2 - yz + 3tu, xyz^2 + z^4, xyt - 3u^3, x^2t^2 + 4y^2u^2) \subseteq P = k[x, y, z, t, u]$.

1) Compute the regularity of I using BettiDiagram
2) Compare regularity and Betti numbers of I with those of $Lt_{revlex}(I)$. Is $Lt_{revlex}(I)$ of Borel type?
3) Compute the regularity of I using $gin(I)$.

A different approach by using a Trung's result

Definition

An element $x \in P_1$ is filter regular for P/I if

$$(P/I)_i \xrightarrow{\cdot x} (P/I)_{i+1}$$

is injective for $i \gg 0$.

Equivalently $x \notin \wp \ \forall \wp \in Ass(I), \ \wp \neq m.$

Hence x is filter regular iff

$$(I : x)_i = I_i \quad \forall i \gg 0 \quad \text{or} \quad \text{equivalently} \quad \lambda(I : x/I) < \infty$$

For example x_n is a filter regular element for an ideal of Borel type.

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For example x_n is a filter regular element for an ideal of Borel type.

Basic idea

Remark. If x is filter regular for P/I, then

$$reg(P/I) = \max\{reg(I : x/I), reg(P/I + (x))\}$$

Definition

$y_1, \ldots, y_t$ is a filter regular sequence for P/I if y_1 is filter regular and y_i is filter regular in $P/(y_1, \ldots, y_{i-1})$ for every $i = 2, \ldots, t$.

- Let $y_1, \ldots, y_t$ be a filter regular sequence for P/I. Then $y_1, \ldots, y_t$ is a s.o.p. in P/I
- If $|k| = \infty$ then there exists a maximal filter regular sequence $y_1, \ldots, y_d$ where $d = \dim P/I$.
- $[I + (y_1, \ldots, y_i) : y_{i+1}]_r = [I + (y_1, \ldots, y_i)]_r \ \forall r \gg 0.$

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Trung's result

Assume $d \geq 1$. Let $\underline{y} := y_1, \ldots, y_d$ be a sequence of linear forms. Define

$$I_i := I_{i-1} + (y_i) \quad (I_0 = I)$$

$$a^i_{\underline{y}}(I) := I_{i-1} : y_i / I_{i-1}$$

If $\lambda(a^i_{\underline{y}}(I)) < \infty$, then

$$reg(a^i_{\underline{y}}(I)) = \sup\{r : \; [I_{i-1} : y_i]_r \neq [I_{i-1}]_r\}$$

with $reg(a^i_{\underline{y}}) := -\infty \quad$ if $\quad I_{i-1} : y_i = I_{i-1}$.

$\underline{y} := y_1, \ldots, y_d$ is a filter-regular sequence for P/I if and only if $\lambda(a^i_{\underline{y}}) < \infty \; \forall i$.

We control the regularity in terms of these integers:

Theorem (Trung)

Let $\underline{y} := y_1, \ldots, y_d$ be a maximal filter regular sequence for P/I. Then

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Let $\underline{x} := x_n, \ldots, x_{n-d+1}$, by the properties of τ=revlex we have

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Bermejo-Gimenez, Trung's algorithm

What is the needed genericity?

- Consider a (*sparse*) change of coordinates
- Compute $a^i_{\underline{x}}(Lt_{revlex}(I))$ where $\underline{x} := x_n, \ldots, x_{n-d+1}$
- The generality is enough if $\underline{x} := x_n, \ldots, x_{n-d+1}$ is a filter regular sequence for $P/Lt_{revlex}(I))$ (equivalently for P/I), that is $\lambda(a^i_{\underline{x}}(Lt_{revlex}(I))) < \infty$
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