Computational Commutative Algebra
Castelnuovo-Mumford regularity

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1 Bounds on the regularity and *Open Problems*

Bounds in terms of the degrees of generators

In the previous lectures we considered two measures of the complexity of an homogeneous ideal $I \subseteq P = k[x_1, \ldots, x_n]$:

- $d(I)$ the maximum degree of a polynomial in a minimal system of generators of I (actually of the generators of $gin_{revlex}(I)$)
- $reg(I)$: the maximum degree of the syzygies in a minimal free resolution of I

Question How much bigger can $reg(I)$ be than $d(I)$?

Obviously:

$$d(I) \leq reg(I)$$

Conjecture (Bayer '82):

$$reg(I) \leq d(I)^{2^{n-1}}$$

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Giusti-Galligo ('84) : If char $k = 0$, then

$$reg(I) \leq (2d(I))^{2^{n-2}}$$

There are examples with very large regularity (Mayr-Mayer).

The regularity can really be doubly exponential in the degrees of the generators and the number of the variables.

Koh ('98) : For each integer $r \geq 1$ there exists an ideal $I_r \subseteq P = k[x_1, \ldots, x_n]$ with $n = 22r$ generated by quadrics such that

$$reg(I_r) \geq 2^{2^{r-1}}$$

These examples are highly non reduced (see also Giaimo's work for a way of making reduced examples).

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Bayer-Mumford in any characteristic

$$reg(I) \leq (2d(I))^{(n-1)!}$$

In the same paper they asked whether Giusti-Galligo's bound holds in any characteristic.

Caviglia-Sbarra: If $\mathrm{ht}(I) = c < n$ and I is generated in degree $\leq d$, then

$$reg(I) \leq (d^c + (d-1)c + 1)^{2^{n-c-1}}$$

As a consequence we may deduce

- $n = 2$ $reg(I) \leq 2d$
- $n \geq 3$ $reg(I) \leq (d^2 + 2d - 1)^{2^{n-3}} \leq (2d)^{2^{n-2}}$ (Giusti-Galligo's bound) (the worst case is $\mathrm{ht}(I) = 2$.)

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Problem: (Peeva-Stillman) Let $d_1 \geq d_2 \geq \dots$ be the degrees of the elements in a minimal system of generators of I. Set $c = ht(I)$, find conditions on I such that

$$\mathrm{reg}(I) \leq d_1 + \cdots + d_c - c + 1$$

Exercise.
Let $I \subseteq P = k[x_1, \dots, x_n]$, $\dim P/I = 0$, I is generated in degree $\leq d$, then

$$reg(I) \leq nd - n + 1$$

Sjögren : The previous fact holds assuming $\dim P/I \leq 1$.

For smooth (or nearly smooth) varieties there are much better bounds, linear in the degrees of the generators and in the number of the variables (see Bertram-Ein-Lazarsfeld and Chardin-Ulrich).

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Eisenbud-Goto Conjecture (84): If $\wp \subseteq (x_1, \ldots, x_n)^2$ is a prime homogeneous ideal, then

$$\operatorname{reg}(P/\wp) \leq e(P/\wp) - n + dimP/\wp$$

- It is proved for irreducible curves (Gruson, Lazarsfeld, Peskine '83)
- It is proved for smooth surfaces (Bayer-Mumford '93). Some more generality (Brodman'99)
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Regularity of the Tangent Cone

Let $A = k[[x_1, \ldots, x_n]]/I$ a local ring and let m be its maximal ideal.
We define the homogeneous k-standard algebra

$$gr_m(A) = \oplus_{n \geq 0} m^n/m^{n+1}$$

which is called the associated graded ring or the tangent cone of A.

Geometric meaning: If A is the localization at the origin of the coordinate ring of an affine variety V passing through 0, then $gr_m(A)$ is the coordinate ring of the *tangent cone* of V, which is the cone composed of all lines that are limiting positions of secant lines to V in 0.

We have the following presentation

$$gr_m(A) \simeq k[x_1, \ldots, x_n]/I^*$$

where I^* is the ideal generated by the initial forms (w.r.t. the m-adic filtration) of the elements of I. The ideal I^* can by computed by using a slight modification of Buchberger's algorithm (see SINGULAR).

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Example

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Consider the power series $A = k[[t^4, t^5, t^{11}]]$. This is a one-dimensional local domain and

$$A = k[[x, y, z]]/I \quad \text{where} \quad I = (x^4 - yz, y^3 - xz, z^2 - x^3y^2).$$

We can prove that

$$gr_m(A) = k[x, y, z]/(xz, yz, z^2, y^4)$$

We have $\dim A = \dim\ gr_m(A) = 1$, but depth $gr_m(A) = 0$.

We always have $\dim A = \dim\ gr_m(A)$, but the above example shows that

$$A \text{ Cohen-Macaulay} \not\Longrightarrow gr_m(A) \text{ Cohen-Macaulay}$$

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Minimal free resolution of the tangent cone

Denote by $\mu(\ \ )$ the minimal number of generators of an ideal of A. The Hilbert function of A is, by definition

$$HF_A(n) := dim_k\ m^n/m^{n+1} = \mu(m^n)$$

for every $n \geq 0$. Hence HF_A is the Hilbert function of the homogeneous k-standard algebra

$$gr_m(A) = \oplus_{n\geq 0} m^n/m^{n+1}$$

In particular $e(A) = e(gr_m(A)),\ \dim A = \dim gr_m(A)$. Several papers have been produced concerning the following problem:

Problem: Compare the numerical invariants of the R-free minimal resolution of A ($R = k[[x_1, \ldots, x_n]]$) with those of the P-free minimal graded resolution ($P = k[x_1, \ldots, x_n]$) of $gr_m(A)$:

$$0 \to R^{\beta_h(I)} \to R^{\beta_{h-1}(I)} \to \cdots \to R^{\beta_0(I)} \to I \to 0$$

$$0 \to P^{\beta_s(I^*)} \to P^{\beta_{s-1}(I^*)} \to \cdots \to P^{\beta_0(I^*)} \to I^* \to 0$$

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Minimal free resolution of the tangent cone

$$\beta_i(I) \leq \beta_i(I^*)$$

In general is $<$ (see R.-Sharifan for more complete information).

Example (Herzog,R., Valla)

Consider $I = (x^3 - y^7, x^2y - xt^3 - z^6)$ in $R = k[[x, y, z, t]]$. Since I is a complete intersection, then a minimal free resolution of I is given by:

$$0 \to R \to R^2 \to I \to 0.$$

But

$$I^* = (x^3, x^2y, x^2t^3, xt^6, x^2z^6, xy^9 - xz^6t^3, xy^8t^3, y^7t^9),$$

hence $\mu(I^*) = 8$ and a minimal free resolution of I^* is given by

$$0 \to P \to P^6 \to P^{12} \to P^8 \to I^* \to 0$$

In particular depth $A = 2$ and depth $gr_m(A) = 0$.

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Regularity of $gr_m(A)$

It is an interesting problem to study the Castelnuovo-Mumford regularity of the tangent cone of a Cohen-Macaulay local ring.

- If $gr_m(A)$ is a Cohen-Macaulay graded algebra, then

$$reg(gr_m(A)) \leq e(A) - n + d$$

- A 1-dimensional Cohen-Macaulay then

$$reg(gr_m(A)) \leq e(A) - 1.$$

Problem. [R., Trung, Valla] Let (A, m) be a local Cohen-Macaulay ring. Is $reg(gr_m(A))$ bounded by a polynomial function (possibly linear) of the multiplicity $e(A)$ and the codimension?

Srinivas-Trivedi, Rossi-Trung-Valla proved very large bounds.

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Regularity of $G = gr_m(A)$

The following results allow to repeat the procedure of Lesson 4 (Mumford's inequality) for studying $reg(G)$.

- Assume that $depthA > 0$. Then

$$reg(G) = \text{g-}reg(G).$$

- Let x be a generic element of $m - m^2$ and $\overline{G} = gr_{m/(x)}(A/(x))$. Then

$$\text{g-}reg(G/(x^*)) = \text{g-}reg(\overline{G}).$$

Theorem (R, Valla,Trung)

Let A be a Cohen-Macaulay local ring with $d = \dim A \geq 1$. Then

(i) $\text{reg}(G) \leq e(A) - 1$ *if* $d = 1$,

(ii) $\text{reg}(G) \leq e(A)^{2((d-1)!)-1}[e(A) - 1]^{(d-1)!}$ *if* $d \geq 2$.

Regularity of $G = gr_m(A)$

The following results allow to repeat the procedure of Lesson 4 (Mumford's inequality) for studying $reg(G)$.

- Assume that $depth A > 0$. Then

$$reg(G) = \text{g-}reg(G).$$

- Let x be a generic element of $m - m^2$ and $\overline{G} = gr_{m/(x)}(A/(x))$. Then

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Finiteness of HF

As in Kleiman's theorem (Lesson 4), as an application of the bound on the Castelnuovo-Mumford regularity, we obtain the finiteness of Hilbert functions of local rings with given dimension and multiplicity.

Theorem (Srinivas, Trivedi; R, Valla, Trung)

Given two positive integers d and q there exist only a finite number of Hilbert functions for a local Cohen-Macaulay ring A with $\dim A = d$ and $e \leq q$.

Local version of Kleiman's Theorem?

We remark that the analogous of Kleiman result does not hold in the local case.

Srinivas and Trivedi showed with the following example that the class of local domains of dimension two and multiplicity 4 does not have a finite number of Hilbert functions. Let

$$A_r := k[[X, Y, Z, T]]/\wp_r$$

where

$$\wp_r = (Z^r T^r - XY, X^3 - Z^{2r} Y, Y^3 - T^{2r} X, X^2 T^r - Y^2 Z^r).$$

Then it is easy to see that $\wp_r$ is a prime ideal and the associated graded ring of A_r is the standard graded algebra

$$G_r = k[X, Y, Z, T]/(XY, X^3, Y^3, X^2 T^r - Y^2 Z^r).$$

We have

$$\mathrm{reg}(G_r) = r + 1, \quad HS_{A_r}(z) = \frac{1 + 2z + 2z^2 - z^{r+2}}{(1 - z)^2}.$$

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