Computational Commutative Algebra
Castelnuovo-Mumford regularity

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Tehran, 2-7 July 2011

Castelnuovo Mumford regularity

The Castelnuovo Mumford regularity

- is one of the most important invariants of a graded module.
- is related to the theory of syzygies which connects the qualitative study of algebraic varieties and commutative rings with the study of their defining equations.
- is related to the local cohomology theory
- is a good measure of the complexity of computing Gröbner bases.
- is a very active area of research which involves specialists working in commutative algebra, algebraic geometry and computational algebra.

Contents

Notations

- Denote

$$P = k[x_1, \ldots, x_n]$$

a polynomial ring over a field k with deg $x_i = 1$

$P_j :=$ k-vector space generated by the forms of P of degree j.

- M a finitely generated graded P-module (such as an homogeneous ideal I or P/I), i.e.

$$M = \oplus_i M_i$$

as abelian groups and $P_j M_i \subseteq M_{i+j}$ for every i, j.

Let $d \in \mathbb{Z}$, the d-th twist of M

$$M(d)_i := M_{i+d}.$$

Hilbert Function

Definition

The numerical function

$$HF_M(j) := \dim_k M_j$$

is called the Hilbert function of M.

Assume $M = P/I$ where I is an homogeneous ideal of P. Then

$$HF_{P/I}(j) = \dim_k (P/I)_j$$

An important motivation arises in projective geometry: let $X \subseteq \mathbb{P}^r$ be a projective variety defined by $I = I(X) \subseteq P = k[x_0, \ldots, x_r]$.

If we write $A(X) = P/I(X)$ for the homogeneous coordinate ring of X :

$$HF_X(j) = \dim_k A(X)_j = \dim_k P_j - \dim_k I_j = \binom{r+j}{r} - \dim_k I_j$$

$\dim_k I_j$ --→ the "number" of hypersurfaces of degree j vanishing on X.

Hilbert Function

Let τ be a term ordering on $\mathbb{T}^n$, then $G = \{f_1, \ldots, f_s\}$ is a τ-Gröbner basis of I if

$$\mathrm{Lt}_\tau(I) := < \mathrm{Lt}_\tau(f) \; : \; f \in I >= \{\mathrm{Lt}_\tau(f_1), \ldots, \mathrm{Lt}_\tau(f_s)\}$$

The residue classes of the elements of $\mathbb{T}^n \setminus \mathrm{Lt}_\tau(I)$ form a k-basis of P/I.

Proposition (Macaulay)

For every $j \geq 0$

$$HF_{P/I}(j) = HF_{P/\mathrm{Lt}_\tau(I)}(j)$$

Hilbert Polynomial, Hilbert Series

- $HF_M(j)$ for $j \gg 0$ agrees with $HP_M(X)$ a polynomial of degree $d-1$ where $d =$ Krull dimension of M (> 0).

  $HP_M(j)$ is called Hilbert Polynomial and it encodes several asymptotic information on M.

- A more compact information can be encoded by the Hilbert Series

$$HS_M(z) := \sum_{j \geq 0} HF_M(j) z^j = \frac{h_M(z)}{(1-z)^d} \quad \text{(Hilbert} - \text{Serre)}$$

  where $h_M(1) = e > 0$ is the multiplicity of M and $d = \dim M$.

- Define

$$\text{reg-index}(M) := \max\{j : \quad HF_M(j) \neq HP_M(j)\}$$

Example

(CoCoA)

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-------------------------------
-- The current ring is R ::= Q[x,y,z];
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I:=Ideal(x^2,xy,xz, y^3);
H:=Hilbert(R/I);
H;
H(0) = 1
H(t) = 3   for t >= 1
-------------------------------
HilbertPoly(R/I);
3
-------------------------------
Poincare(R/I);   (or HilbertSeries(R/I);)
(1 + 2x) / (1-x)
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Minimal free resolutions

- A graded free resolution of M as a graded P-module is an exact complex ($\ker f_{j-1} = \operatorname{Im} f_j$ for every j)

$$\mathbb{F}: \quad \ldots F_h \xrightarrow{f_h} F_{h-1} \xrightarrow{f_{h-1}} \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

  where F_i are free P-modules and f_i are homogeneous homomorphisms (of degree 0).
- $\mathbb{F}$ is minimal if for every $i \geq 1$

$$\operatorname{Im} f_i \subseteq mF_{i-1}$$

  where $m = (x_1, \ldots, x_n)$.

Existence of minimal graded free resolutions

- Every finitely generated P-module admits a minimal free resolution:

$$\mathbb{F}: \quad \ldots F_h \xrightarrow{f_h} F_{h-1} \xrightarrow{f_{h-1}} \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

- We are interested in building a graded minimal P-free resolution:

$M =< m_1, \ldots, m_t >_P$ minimally generated with $\deg m_j = a_{0j}$.
Define the homogeneous epimorphism:

$$F_0 = \oplus_j P(-a_{0j}) \xrightarrow{f_0} M \to 0$$

$$e_j \quad \rightsquigarrow \quad m_j$$

By the minimality of the system of generators

$$\text{Ker } f_0 \subseteq mF_0$$

We can iterate the procedure

$$0 \to \text{Ker } f_i \to F_i = \oplus_i P(-a_{ij}) \xrightarrow{f_i} Kerf_{i-1} \to 0$$

Example

$I = (x^2, xy, xz, y^3)$ in $P = k[x,y,z]$. Define

$$P(-2)^3 \oplus P(-3) \xrightarrow{f_0} I \to 0$$

$$\begin{aligned} e_1 &\rightsquigarrow x^2 \\ e_2 &\rightsquigarrow xy \\ e_3 &\rightsquigarrow xz \\ e_4 &\rightsquigarrow y^3 \end{aligned}$$

$Syz_1(I) = \text{Ker } f_0$ is generated by $s_1 = ye_1 - xe_2$; $s_2 = ze_1 - xe_3$; $s_3 = ze_2 - ye_3$; $s_4 = y^2e_2 - xe_4$. Define

$$P(-3)^3 \oplus P(-4) \xrightarrow{f_1} Syz_1(I) \to 0$$

$$e_i' \rightsquigarrow s_i$$

$Syz_2(I) = \text{Ker } f_1$ is generated by $s = ze_1' - ye_2' + xe_3'$.
A minimal free resolution of I as P-module is given by:

$$0 \to P(-4) \xrightarrow{f_2} P(-3)^3 \oplus P(-4) \xrightarrow{f_1} P(-2)^3 \oplus P(-3) \xrightarrow{f_0} I \to 0.$$

$$1 \rightsquigarrow s$$

Minimal graded free resolution

A minimal graded free resolution of M as P-module can be presented as follows:

$$\mathbb{F}: \quad \cdots \oplus_{j=1}^{\beta_h} P(-a_{hj}) \xrightarrow{f_h} \oplus_{j=1}^{\beta_{h-1}} P(-a_{h-1j}) \xrightarrow{f_{h-1}} \dots \xrightarrow{f_1} \oplus_{j=1}^{\beta_0} P(-a_{0j}) \xrightarrow{f_0} M \to 0$$

It will be useful rewrite the resolution as follows:

$$\cdots \to F_i = \oplus_{j\geq 0} P(-j)^{\beta_{ij}} \to \cdots \to \oplus_{j\geq 0} P(-j)^{\beta_{0j}} \to M$$

1) $\beta_{ij} \geq 0$
2) $\beta_{ij} =$ cardinality of the shift $(-j)$ in position i ($\beta_i = \sum \beta_{ij}$)

Question. Does β_{ij} (hence a_{ij}) depend on the maps f_i of the resolution?

We remind that in proving the existence of a minimal free resolution we can choose different system of generators of the kernels, hence different maps.

Basic facts

We prove that the graded Betti numbers are uniquely determined by M.

Proposition

$$\beta_{ij} = \beta_{ij}(M) = dim_k Tor_i^P(M,k)_j$$

and we call these integers graded Betti numbers of M.

In fact

$$Tor_i^P(M,k) = H_i(\mathbb{F} \otimes P/m)$$

By the minimality of $\mathbb{F}$ the maps of the new complex $\mathbb{F} \otimes P/m$ are trivial, hence we have

$$Tor_i^P(M,k)_j = [\oplus_{m\geq 0} P(-m)^{\beta_{im}} \otimes P/m]_j = [\oplus_{m\geq 0} k(-m)^{\beta_{im}}]_j =$$

$$= \oplus_{m\geq 0}(k_{j-m})^{\beta_{im}} \underset{m=j}{=} k^{\beta_{ij}}$$

The resolution fixes the Hilbert Function

Let I be an homogeneous ideal of P.

Proposition

If $\beta_{ij} = \beta_{ij}(P/I)$ are the graded Betti numbers of P/I, then the Hilbert series of P/I is given by

$$HS_{P/I}(z) = \frac{1 + \sum_{ij}(-1)^{i+1}\beta_{ij}z^j}{(1-z)^n}$$

If we consider the previous example $I = (x^2, xy, xz, y^3)$ in $P = k[x, y, z]$. We have seen that a minimal free resolution of I as P-module is given by:

$$0 \to P(-4) \to P(-3)^3 \oplus P(-4) \to P(-2)^3 \oplus P(-3) \to P \to P/I \to 0.$$

Since $HS_{P(-d)^\beta}(z) = \frac{\beta z^d}{(1-z)^n}$, then

$$HS_{P/I}(z) = \frac{1 - 3z^2 - z^3 + 3z^3 + z^4 - z^4}{(1-z)^3} = \frac{1+2z}{1-z}$$

Exercise

Consider

$$X = \{P_1, \dots, P_4\} \subseteq \mathbb{P}^2$$

four distinct points in the plane.

Denote $A(X) = k[x_0, x_1, x_2]/I(X)$ the corresponding coordinate ring. Prove:

- the Hilbert polynomial of a set of four points, no matter what the configuration, is a constant polynomial $HP_X(n) = 4$.
- the Hilbert function of X depends only on whether all four points lie on a line.
- The graded Betti numbers of the minimal resolution, in contrast, capture all the remaining geometry: they tell us whether any three of the points are collinear as well.

Hilbert's Syzygy Theorem

Theorem (Hilbert's Syzygy Theorem)

Every finitely generated P-module has a ***finite*** *free resolution (of length $\leq n$)*

We remind that $Tor_i(k, M) = H_i(\mathbb{K} \otimes M)$ where $\mathbb{K}$ is a minimal free resolution of $k = P/(x_1, \ldots, x_n)$ as P-module.

Hence we consider the Koszul complex of $(x_1, \ldots, x_n)$.:

$$\mathbb{K} : 0 \to P(-n)^{\binom{n}{n}} \to P(-n+1)^{\binom{n}{n-1}} \to \cdots \to P(-1)^{\binom{n}{1}} \to P$$

We deduce

$$Tor_i(k, M) = H_i(\mathbb{K} \otimes M) = 0$$

for every $i \geq n+1$ ($K_i = 0$ for $i \geq n+1$).

Auslander-Buchsbaum formula

If M has the following minimal P-free resolution:

$$0 \to F_h = \oplus_{j \geq 0} P(-j)^{\beta_{hj}} \to \cdots \to \oplus_{j \geq 0} P(-j)^{\beta_{0j}} \to M$$

Define the Projective dimension (or Homological dimension) of M

$$pd(M) := \max\{i : \beta_{ij}(M) \neq 0 \text{ for some } j\}$$

that is $h =$ length of the resolution.

Theorem (Auslander-Buchsbaum)

$$pd_P(M) = n - \mathrm{depth}(M)$$

where $\mathrm{depth}(M) =$ *length of a (indeed any) maximal* M*-regular sequence in* $m = (x_1, \ldots, x_n)$.

M is Cohen-Macaulay $\iff$ $\mathrm{depth} M = \dim M$ $\iff$ $\mathrm{pd}_P(M) = n - \dim M$.

Betti Diagram

The numerical invariants in a minimal free resolution can presented by using "a piece of notation" introduced by Bayer and Stillman: the Betti diagram.

This is a table displaying the numbers β_{ij} in the pattern

	0	1	2	...	i
0 :	β_{00}	β_{11}	β_{22}	$\cdots$	β_{ii}
1 :	β_{01}	β_{12}	β_{23}	$\cdots$	β_{ii+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s :	β_{0s}	β_{1s+1}	β_{2s+2}	$\cdots$	β_{ii+s}
$\sum$	β_0	β_1	β_2	$\cdots$	β_i

with β_{ij} in the i-th column and $(j-i)$-th row.

Thus the i-th column corresponds to the i-th free module

$$F_i = \oplus_j P(-j)^{\beta_{ij}}.$$

Example

(CoCoA)

```
  Use R ::= QQ[t,x,y,z];
  I := Ideal(x^2-yt,xy-zt,xy);
  Res(I);
0 --> R^2(-5) --> R^4(-4) --> R^3(-2)
-------------------------------
  BettiDiagram(I);
         0    1    2
 --------------------
  2:     3    -    -
  3:     -    4    2
 --------------------
 Tot:    3    4    2
 ------------------------------
```

Definition

Given a minimal P-free resolution of M :

$$\mathbb{F} : ..... \qquad \to F_i = \oplus P(-j)^{\beta_{ij}(M)} \to \cdots \to F_0 = \oplus P(-j)^{\beta_{0j}(M)}$$

the Castelnuovo-Mumford regularity of M

$$reg(M) = \max_i \{j - i : \beta_{ij}(M) \neq 0\}$$

Equivalently if we write

$$\mathbb{F} : \quad \cdots \oplus_{j=1}^{\beta_h} P(-a_{hj}) \xrightarrow{f_h} \oplus_{j=1}^{\beta_{h-1}} P(-a_{h-1j}) \xrightarrow{f_{h-1}} \ldots \xrightarrow{f_1} \oplus_{j=1}^{\beta_0} P(-a_{0j}) \xrightarrow{f_0} M \to 0$$

Define

$$a_i := \max_j \{a_{ij} - i\} (\geq 0)$$

then

$$reg(M) = \max_i \{a_i\}$$

If we consider THE example

$$I = (x^2, xy, xz, y^3) \subseteq P = k[x, y, z].$$

We have seen that a minimal free resolution of I as P-module is given by:

$$0 \to F_2 = P(-4) \xrightarrow{f_2} F_1 = P(-3)^3 \oplus P(-4) \xrightarrow{f_1} F_0 = P(-2)^3 \oplus P(-3) \xrightarrow{f_0} I \to 0.$$

Then

- $pd(I) = 2$
- $reg(I) = 3 =$ max degree of a minimal generator.
- $\dim P/I = 1$ (we know that $HS_{P/I}(z) = \frac{1+2z}{1-z}$).

  Hence P/I is not Cohen-Macaulay since $pd(P/I) = 3 > 3 - \dim P/I = 2$.
- $reg\text{-}index(P/I) = 1 < reg(P/I) = 2$

```
 Use P ::= Q[x,y,z,w];
 I := Ideal(xz-yw, xw-y^2,  x^2y+xzw, xy^2, xyz);
 Reg(I);
 4
 -------------------------------
 Res(I);
-------------------------------
P^2(-7) -> P^6(-6) -> P^5(-4)(+)P^3(-5)-> P^2(-2)(+)P^3(-3)
-------------------------------
BettiDiagram(I);
-------------------------------
        0    1    2    3
-------------------------
 2:     2    -    -    -
 3:     3    5    -    -
 4:     -    3    6    2
-------------------------
Tot:    5    8    6    2
-------------------------------
```

Remarks

- (Exercise) If M has finite length, then $reg(M) = \max\{j \ : \ M_j \neq 0\}$.
- $\mathrm{reg}(I) = \mathrm{reg}(P/I) + 1 \geq$ maximum degree of a minimal generator of I
- $\mathrm{reg}(P/I)$ **coincides with the last non-zero row in the Betti diagram**

Stillman's conjecture

$R = k[x_1, ..., x_n]/I = (f_1, \dots, f_r)$ where f_i are forms of degree d_i.

In general $\mathrm{pd}(I)$, as well $\mathrm{reg}(I)$, can grew relatively fast as one increases the number of generators and the degrees.

Conjecture (Stillman)

There is an upper bound, independent of n, on $pd(I)$, for any ideal I generated by r homogeneous polynomials of given degrees.

Ananyan-Hochster (2011): Positive answer if $d_i \leq 2$.

Equivalently to:

Conjecture (Caviglia-Kumini)

There is an upper bound, independent of n, on the Castelnuovo-Mumford regularity for any ideal I generated by r homogeneous polynomials of given degrees.

Exercises

Exercise Consider the homogeneous coordinate ring of the "twisted cubic":

$$R = K[s^3, s^2t, st^2, t^3]$$

- Prove that $R = P/I$ where $P = K[x_0, \dots, x_3]$ and $I = I_2\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$
- Prove that R is CM
- Compute $\mathrm{HF}_R(j)$, $\mathrm{reg}(R)$
- Compare $\mathrm{reg}(I)$ and $\mathrm{reg}(Lt_\tau(I))$ with τ any term ordering

Exercise Consider the homogeneous coordinate ring of the smooth rational quartic in $\mathbb{P}^3$

$$R = K[s^4, s^3t, st^3, t^4]$$

- Prove that $R \simeq P/I$ where $P = K[x_0, \dots, x_3]$ and $I = I_2\begin{pmatrix} x_0 & x_1^2 & x_1x_3 & x_2 \\ x_1 & x_0x_2 & x_2^2 & x_3 \end{pmatrix}$
- Prove that R is not CM
- Compute $\mathrm{reg}(I)$

Contents

Alternative definitions

One of the aspects that makes the regularity very interesting is that $reg(M)$ can be computed in different ways.

- We say that M is m-regular for some integer m if

$$reg(M) \leq m$$

$(reg(M) := \min\{m : \ M \text{ is } m\text{-regular } \})$ Hence

$$M \text{ is } m\text{-regular} \iff \beta_{ij}(M) = 0 \quad \forall j \geq i + m + 1$$

(equivalently $Tor_i^P(M,k)_j = 0 \quad \forall j \geq i + m + 1$).

Alternative definitions

Let $\mathbb{F} = \{F_i\}$ be a graded minimal free resolution of M.

$$M \text{ is } m\text{-regular} \implies F_i \text{ has no generators in degrees } \geq m+i+1$$

Consider $Hom(\mathbb{F}, P)$ and denote $F_i^* = Hom_P(F_i, P)$, then

$$M \text{ is } m\text{-regular} \implies [F_i^*]_{\leq -m-i-1} = 0$$

$$\downarrow$$

$$Ext_P^i(M, P) = H_i(Hom(\mathbb{F}, P))$$

Regularity in terms of Ext's

$$reg(M) := \min\{m : \; Ext^i_P(M,P)_j = 0 \; : \forall j \leq -m-i-1\}$$

The above equality is hard to apply because in principle infinitely many conditions must be checked. We introduce a new definition given by Mumford for sheaves:

M is weakly m-regular if for every i

$$Ext^i_P(M,P)_{-m-i-1} = 0$$

If either $depthM > 0$ or $M = P/I$ then

$$reg(P/I) := \min\{m : \; Ext^i_P(P/I,P)_{-m-i-1} = 0\;\}$$

In terms of the Local Cohomology

Denote by $H^i_m(M)$ the local cohomology module with support in m, $0 \leq i \leq d = \dim M$.

By using the local duality (Eisenbud, A 4.2)

$$H^i_m(M)_j \simeq Ext_P^{n-i}(M, P)_{-j-n}$$

We recall that $H^i_m(M)$ are Artinian and we let

$$end(H^i_m(M)) := \max\{j : \ H^i_m(M)_j \neq 0\}$$

$(\max 0 = -\infty)$

$$reg(M) = \max\{end(H^i_m(M)) + i \ : \ 0 \leq i \leq d\}$$

In terms of the Local Cohomology

By Grothendieck-Serre's formula (Bruns-Herzog Theor. 4.4.3)

$$HP_M(i) - HF_M(i) = \sum_{j=0}^{d} (-1)^{j+1} \lambda(H^j_m(M)_i)$$

As a consequence

$$HP_M(i) = HF_M(i) \quad \forall i > reg(M)$$

$$reg\text{-}index(M) \leq reg(M)$$

Regularity and exact sequences

This approach gives a quite easy proof of the following

Proposition

Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of graded finitely generated P-modules (homogeneous maps), then

1) $reg(A) \leq \max(reg(B), reg(C) + 1)$
2) $reg(B) \leq \max(reg(A), reg(C))$
3) $reg(C) \leq \max(reg(A) - 1, reg(B))$
4) *If A has finite length, then* $reg(B) = \max(reg(A), reg(C))$.

Hint: consider the long exact sequence

$$\cdots \to Ext^{j-1}(A, P) \to Ext^{j}(C, P) \to Ext^{j}(B, P) \to$$

$$\to Ext^{j}(A, P) \to Ext^{j+1}(C, P) \to \dots$$

Regularity and linear resolutions

Definition

I has a d-linear resolution if it is generated in one degree, say d, and $\beta_{ij}(I) = 0$ for all $j \neq i + d$. If this is the case

$$reg(I) = d.$$

$$0 \to P^{\beta_h}(-d-h) \to \cdots \to P^{\beta_1}(-d-1) \to P^{\beta_0}(-d) \to I \to 0$$

The matrices associated to the maps of the resolution have linear entries (or zero).

Regularity and linear resolution

Proposition

Set $I_{\geq j} := I \cap m^j$.

$$r = reg(I) \implies I_{\geq j} \text{ has } j\text{-linear resolution } \forall j \geq r$$

Important steps: • $I_{<r>}$ has r-linear resolution

• M has d-linear resolution $\implies$ mM has $(d+1)$-linear resolution.

It is enough to consider the exact sequence of graded modules

$$0 \to mM \to M \to M/mM \to 0$$

Then by the exact sequence

$$reg(mM) \leq \max\{reg(M), reg(M/mM) + 1\} = \max\{d, d+1\}$$

On the other hand $reg(mM) \geq d + 1 = indeg\,(mM)$.

```
Use P ::= Q[x,y,z];
I := Ideal(x^2,xy,xz,y^3);
Reg(I);
3
-------------------------------
Res(I);
0 --> P(-4) --> P^3(-3)(+)P(-4) --> P^3(-2)(+)P(-3)
-------------------------------

J:=Intersection(I,Ideal(x,y,z)^3);

Res(J);
-------------------------------
0 --> P^3(-5) --> P^9(-4) --> P^7(-3)
```

Regularity and hyperplane sections

Let $F \in P$ be homogeneous such that $0 :_M F$ has finite length, by using the comparison between regularities in exact sequences, we get

$$reg(M) = \max(reg(0 :_M F), reg(M/FM) - \deg F + 1)$$

(Actually it is enough $\dim(0 :_M F) \leq 1$)

- If $L \in P_1$ is M-regular, then

$$reg(M) = reg(M/LM)$$

- If L is a linear *filter regular element* ($M_n \stackrel{\cdot L}{\to} M_{n+1}$ injective $n \gg 0$)

$$reg(M) = \max\{reg(0 : L), reg(M/LM)\} \geq reg(M/LM)$$

(e.g. $\dim M > 0$, $|K| = \infty$ and L a generic linear form)

Regularity of a CM module

Proposition

Let M be a Cohen-Macaulay graded finitely generated P-modules of dimension d

1) $reg(M) = \deg(h_M(z))$ *where $h_M(z)$ is the h-polynomial of M ($HS_M(z) = \frac{h_M(z)}{(1-z)^d}$)*
2) $reg(M) = reg\text{-}index(M) + d$

Proof: ($|k| = \infty$) Let $J = (L_1, \dots, L_d) \subseteq P$ the ideal generated by a maximal M-regular sequence of linear forms. We know that

$$reg(M) = reg(M/JM)$$

Now M/JM is an Artinian module and

$$reg(M/JM) = \max\{n : (M/JM)_n \neq 0\} = \deg(HS_{M/JM}(z)) = \deg(h_M(z))$$

since $HS_M(z) = \frac{HS_{M/JM}(z)}{(1-z)^d}$. Hence $reg(M) = reg\text{-}index(M/JM) = reg\text{-}index(M) + d$.

Regularity and sums, product, intersection of ideals

Let I, J homogeneous ideals, there are the following exact sequences:

$$0 \to P/I \cap J \to P/I \oplus P/J \to P/I + J \to 0$$

$$0 \to I \cap J/IJ \to P/IJ \to P/I \cap J \to 0$$

We can prove

Theorem

If $(I \cap J)/IJ$ is a module of dimension at most 1, then

1) $reg(I + J) \leq reg(I) + reg(J) - 1$
2) $reg(I \cap J) \leq reg(I) + reg(J)$
3) $reg(IJ) \leq reg(I) + reg(J)$.

Regularity and sums, product, intersection of ideals

- G. Caviglia gave an example with $\dim(I \cap J)/IJ = 2$ and $reg(I + J) \geq reg(I) + reg(J)$

The possibility of extending 2) and 3) to any number of ideals is still unclear.

- Conca and Herzog: If $I_1, \ldots, I_r$ are generated by linear forms, then

$$reg(I_1 \cdots I_r) = \sum_i reg(I_i) = r$$

- Derksen and Sidman: If $I_1, \ldots, I_r$ are generated by linear forms, then

$$reg(I_1 \cap \cdots \cap I_r) = \sum_i reg(I_i) = r$$

- Chardin, Cong, Trung: If $I_1, \ldots, I_r$ are monomial complete intersection ideals , then

$$reg(I_1 \cap \cdots \cap I_r) \leq \sum_i reg(I_i)$$

Powers

I homogeneous ideal, q a positive integer:

$$reg(I^q)?$$

I. Swanson: There exists D such that for every $q \geq 1$

$$reg(I^q) \leq q\ D$$

but she could not provide an estimate.

T.Geramita, A.Gimigliano, Pittelloud: Assume $\text{depth}\ P/I^q \geq \dim P/I - 1,$ then

$$reg(I^q) \leq q\ reg(I)$$

The previous assumption is essential:

Sturmfels, Terai: example with $reg(I^2) > 2reg(I)$

Powers

The problem of bounding $reg(I^q)$ is also related to the regularity of

$$\mathcal{R}(I) = \oplus_q I^q$$

This problem seemed to be hard. So it came as a surprise the following result

Theorem (Cutkosky, Herzog, Trung; Hoa, Herzog, Trung)

Let $d(I)$ denote the maximum degree of I

- $\exists\ e \in \mathbf{N} \quad : \quad reg(I^q) \leq q\ d(I) + e$ *for every* $q \geq 1$.
- $\exists\ e \in \mathbf{N}$ *and* $c \leq d(I) \quad : \quad reg(I^q) = c\ q + e$ *for every* $q \gg 0$.

More precise results are provided assuming that I is generated in the same degree.

Exercises

For monomial ideals, there are some more results in terms of better understood invariants of I.

Exercise 1. Let $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ with $m \leq n$ and

$$I = (x_1^{a_1}, \ldots, x_m^{a_m}).$$

Then

$$reg(I) = a_1 + \cdots + a_m - m + 1.$$

Exercise 2. Under the above assumptions:

- $reg(I^q) = qa_1 + a_2 + \cdots + a_m - m + 1$
- $reg(I^q) \leq q\, reg(I)$ and the equality holds iff $a_2 = \cdots = a_m = 1$.

Regularity of the radical

Ravi proved that if I is a monomial ideal, then

$$reg(\sqrt{I}) \leq reg(I)$$

Problem. Find different classes of ideals for which $reg(\sqrt{I}) \leq reg(I)$.

Chardin-D'Cruz produced examples where $reg(\sqrt{I})$ is the cube of $reg(I)$.

Problem.(Peeva-Stillman) Is $reg(\sqrt{I})$ bounded by a (possibly polynomial) function of $reg(I)$?

Example

Example. [Chardin-D'Cruz] Let n, m be positive integers and let

$$I_{m,n} = (x^m t - y^m z, z^{n+2} - xt^{n+1}) \subseteq K[x, y, z, t]$$

The following equalities hold

1. $\mathrm{reg}(I_{m,n}) = m + n + 2$ (complete intersection)
2. $\mathrm{reg}(\sqrt{I_{m,n}}) = m \cdot n + 2$

Contents

$P = k[x_1, \ldots, x_n],\ |k| = \infty,\ \tau$ a monomial order

$$reg(I) \quad \longleftrightarrow \quad reg(Lt_\tau(I))$$

- I and $Lt_\tau(I)$ have the same Hilbert function
- $\beta_{ij}(I) \leq \beta_{ij}(Lt_\tau(I)) \leq \beta_{ij}(Lex(I))$ (Bigatti, Hulett, Pardue)
- $\beta_{ij}(I) \leq \beta_{ij}(gin_{revlex}(I)) \leq \beta_{ij}(gin_\tau(I))$ (Conca)

  As a consequence

- $reg(I) \leq reg(Lt_\tau(I))$

(usually <)

Properties of $\tau =$ RevLex

Let I be an homogeneous ideal in $P = k[x_1, \ldots, x_n]$, $\tau =$ RevLex

- $F \in P$ homogeneous

$$Lt_\tau(F) \in (x_s, \ldots, x_n),\ 1 \leq s \leq n \implies F \in (x_s, \ldots, x_n)$$

- $Lt_\tau(I + (x_n)) = Lt_\tau(I) + (x_n)$
- $Lt_\tau(I : x_n) = Lt_\tau(I) : x_n$
- $x_n, \ldots, x_s$ is a P/I-regular sequence $\iff$ $x_n, \ldots, x_s$ is a $P/Lt_\tau(I)$-regular sequence

Properties of Borel type ideals

Let I be a Borel type ideal in $P = k[x_1, \dots, x_n]$:

- For any $j = 1, \dots, r$
$$I : x_j^\infty = I : (x_1, \dots, x_j)^\infty$$
(weakly stable, nested)

  or equivalently
- If $\mathcal{P}$ is an associated prime of I, then $\mathcal{P} = (x_1, \dots, x_j)$ for some j.

Hence if I is of Borel type and $\dim P/I > 0$, then $I : x_n / I$ is of finite length.

Gin(I): the generic initial ideal

For a generic $g \in GL_n(K)$, $Lt_\tau(g(I))$ is *constant:*

Theorem (Galligo, Bayer-Stilmann)

There exists $U \neq \emptyset$ a Zariski-open subset of $GL_n(k)$ such that

$$Lt_\tau(g(I)) = Lt_\tau(h(I))$$

for every $g, h \in U$.

Set

$$gin_\tau(I) := Lt_\tau(g(I)) \quad \textit{for every } g \in U$$

$gin_\tau(I)$ is a Borel fixed ideal, in particular of Borel type !!!

Bayer-Stilman's Theorem

Theorem (Bayer-Stilman)

Let $I \subseteq P$ be an homogeneous ideal, $|k| = \infty$, $\tau =$ revlex.

$$reg(P/I) = reg(P/gin_\tau(I))$$

We give here an easy proof. First underline the crucial points:

- $gin_\tau(I) = Lt_\tau(g(I))$ with $\tau =$ revlex

  Assume $d > 0$, by using the properties of $\tau =$ revlex (!!!):

- $gin_\tau(I : x_n) = gin_\tau(I) : x_n \qquad gin_\tau(I + (x_n)) = gin_\tau(I) + (x_n)$.
- since gin_τ is of Borel type $gin_\tau(I) : x_n / gin_\tau(I)$ has finite length (if an associated prime contains x_n, it is the maximal ideal).
- $\dim P/I + (x_n) = \dim P/gin_\tau(I + (x_n)) = \dim P/gin_\tau(I) + (x_n) = d - 1$.

Bayer-Stilman's Theorem

Proof:

- The result is clear if $d = dim P/I = 0$ because they have the same HF.
- Assume $d > 0$, by induction on d :

$$\mathrm{reg}(P/I + (x_n)) = \mathrm{reg}(P/gin_\tau(I + (x_n))).$$

Hence $reg(P/I + (x_n)) = reg(P/gin_\tau(I + (x_n))) = reg(P/gin_\tau(I) + (x_n))$.

We claim:

$$reg(I : x_n/I) = reg(gin_\tau(I) : x_n/gin_\tau(I))$$

In fact $gin_\tau(I : x_n) = gin_\tau(I) : x_n$ and we deduce that they have the same Hilbert function and of finite length.

Then

$$reg(P/I) = \max\{reg(I : x_n/I), reg(P/I + (x_n))\} =$$

$$= \max\{reg(gin_\tau(I) : x_n/gin_\tau(I)), reg(P/gin_\tau(I) + (x_n))\} = reg(P/gin_\tau(I)).$$

A generalization of Bayer-Stilman's Theorem

Theorem

Let $I \subseteq P$ be an homogeneous ideal, $|k| = \infty$, $\tau =$ revlex. Assume that $Lt_\tau(I)$ is of Borel type, then

$$reg(P/I) = reg(P/Lt_\tau(I))$$

Bayer-Charalambous-Popescu proved a refinement of Bayer-Stilman's Theorem (extremal Betti numbers)($\rightarrow$ Juergen's lessons).
From the above extension of Bayer-Stilman's Theorem, it is thinkable that in other situations initial ideals of Borel type could replace gin. This would be appreciated from the computational point of view.

Bayer-Stilman's Theorem in char $k = 0$

Theorem (Bayer-Stilmann; Eliahou-Kervaire)

Let $I \subseteq P$ be an homogeneous ideal, char $k = 0$ $\tau =$ revlex.

$$reg(I) = reg(gin_\tau(I)) = \text{ max degree of a generator of } gin_\tau(I)$$

It can be deduced from the following facts :

- char $k = 0 \implies gin_\tau(I)$ is a strongly stable monomial ideal

(i.e. for any monomial m, $x_i m \in J \implies x_j m \in J, \forall j \leq i$)

- By Eliahou-Kervaire's resolution of stable ideals J

Bayer-Charalambous-Popescu proved a refinement of Bayer-Stilman's Theorem (extremal Betti numbers)($\rightarrow$ Juergen's lessons)

Exercise

Exercise. Consider the homogeneous ideal:
$I = (x^2 - yz + 3tu, xyz^2 + z^4, xyt - 3u^3, x^2t^2 + 4y^2u^2) \subseteq P = k[x, y, z, t, u].$

1) Compute the regularity of I using BettiDiagram
2) Compare regularity and Betti numbers of I with those of $Lt_{revlex}(I)$. Is $Lt_{revlex}(I)$ of Borel type?
3) Compute the regularity of I using $gin(I)$.

A different approach by using a Trung's result

Definition

An element $x \in P_1$ is filter regular for P/I if

$$(P/I)_i \stackrel{\cdot x}{\to} (P/I)_{i+1}$$

is injective for $i \gg 0$.
Equivalently $x \notin \wp \ \forall \wp \in Ass(I), \ \wp \neq m$.

Hence x is filter regular iff

$$(I : x)_i = I_i \quad \forall i \gg 0 \ \text{ or } \ \text{ equivalently } \quad \lambda(I : x/I) < \infty$$

For example x_n is a filter regular element for an ideal of Borel type.

Basic idea

Remark. If x is filter regular for P/I, then

$$reg(P/I) = \max\{reg(I : x/I), reg(P/I + (x))\}$$

Definition

$y_1, \ldots, y_t$ is a filter regular sequence for P/I if y_1 is filter regular and y_i is filter regular in $P/(y_1, \ldots, y_{i-1})$ for every $i = 2, \ldots, t$.

- Let $y_1, \ldots, y_t$ be a filter regular sequence for P/I. Then $y_1, \ldots, y_t$ is a s.o.p. in P/I
- If $|k| = \infty$ then there exists a maximal filter regular sequence $y_1, \ldots, y_d$ where $d = \dim P/I$.
- $[I + (y_1, \ldots, y_i) : y_{i+1}]_r = [I + (y_1, \ldots, y_i)]_r \;\; \forall r \gg 0.$

Trung's result

Assume $d \geq 1$. Let $\underline{y} := y_1, \ldots, y_d$ be a sequence of linear forms. Define

$$I_i := I_{i-1} + (y_i) \quad (I_0 = I)$$

$$a^i_{\underline{y}}(I) := I_{i-1} : y_i / I_{i-1}$$

If $\lambda(a^i_{\underline{y}}(I)) < \infty$, then

$$reg(a^i_{\underline{y}}(I)) = \sup\{r : \ [I_{i-1} : y_i]_r \neq [I_{i-1}]_r\}$$

with $reg(a^i_{\underline{y}}) := -\infty \quad$ if $\quad I_{i-1} : y_i = I_{i-1}$.

$\underline{y} := y_1, \ldots, y_d$ is a filter-regular sequence for P/I if and only if $\lambda(a^i_{\underline{y}}) < \infty \ \forall i$.

We control the regularity in terms of these integers:

Theorem (Trung)

Let $\underline{y} := y_1, \ldots, y_d$ be a maximal filter regular sequence for P/I. Then

$$reg(P/I) = \max\{reg(a^i_{\underline{y}}(I)) : \ 1 \leq i \leq d; reg(P/I_d)\}$$

A further generalization

Let $\underline{x} := x_n, \ldots, x_{n-d+1}$, by the properties of τ=revlex we have

$$reg(a^i_{\underline{x}}(I)) = reg(a^i_{\underline{x}}(Lt_\tau(I)))$$

Theorem

Let $\underline{x} := x_n, \ldots, x_{n-d+1}$. If $\lambda(a^i_{\underline{x}}(Lt_\tau(I))) < \infty \ \forall i$, then

$$reg(P/I) = reg(P/Lt_\tau(I))$$

Bermejo-Gimenez, Trung's algorithm

What is the needed genericity?

- Consider a (*sparse*) change of coordinates
- Compute $a^i_{\underline{x}}(Lt_{revlex}(I))$ where $\underline{x} := x_n, \ldots, x_{n-d+1}$
- The generality is enough if $\underline{x} := x_n, \ldots, x_{n-d+1}$ is a filter regular sequence for $P/Lt_{revlex}(I))$ (equivalently for P/I), that is $\lambda(a^i_{\underline{x}}(Lt_{revlex}(I))) < \infty$
- If $\lambda(a^i_{\underline{x}}(Lt_{revlex}(I))) < \infty$ then

$$reg(P/I) = reg(P/Lt_{revlex}(I))$$

I suggest the tutorial by Dr. Eduardo Saenz de Cabezon in CoCoA School (2009):
http://cocoa.dima.unige.it/conference/cocoa2009/notes/saenz3.pdf

Contents

Historical Notes

Castelnuovo (1893): The germ of the idea of regularity as a special case of "Base-point free pencil trick" (exercises 17.18 and 20.21 in Eisenbud's book).

Zarisky (1960) taught to his students (included Mumford and Kleiman) Castelnuovo's idea.

Mumford (1966): gave a definition of regularity for sheaves in $\mathbf{P}^n$ which is related to the notion of weakly m-regularity given in Lesson 2.

Kleiman's thesis (1965), see also Grothendieck's volume SGA 6 (1970): the notion of regularity is used in the construction of bounded families of ideals with given Hilbert polynomial, a crucial point in the construction of Hilbert or Picard scheme.

Reg-limited

Let $I \subseteq P = k[x_1, \ldots, x_n]$ be an homogeneous ideal.

The most important invariants from the Hilbert polynomial are:

d=Krull dimension, e=multiplicity

- If $A = P/I$ is Cohen-Macaulay, we have seen that $reg(A) =$degree of the h-polynomial $h_A(z)$. Hence (we may assume $I \subseteq m^2$)

$$reg(A) \leq e - n + d$$

  In particular $=$ holds $\iff$ $h_A(z) = 1 + (n-d)z + \cdots + z^{e-n+d}$.

- The following example shows that in general the regularity cannot be bounded by a function $F(e, d, n)$.

  Example. Let $r \in \mathbb{N}^*$ and consider $A = k[x, y]/(x^2, xy^r)$ which is 1-dimensional non C-M. In this case $e(A) = 1$ but $reg(A) = r$.

  Geometric information can produce better situations.

Reg-limited

- (Castelnuovo) $I = I(C)$ where C is a smooth curve:

$$reg(I) \leq e - 1$$

More in general

- (Gruson-Lazarsfeld-Peskine) $k = \overline{k}$, $I = I(C)$ where C is a reduced irreducible curve in $\mathbb{P}^n$:

$$reg(I) \leq e - n + 2$$

Here we will present an algebraic version of a result by Kleiman (1971) in the case of equidimensional reduced schemes.

The problem is related to the finitness of Hilbert functions for classes of graded k-algebras with given multiplicity.

Reg-limited

Let $\mathcal{C}$ be a class of homogeneous ideals in $P = k[x_1, \dots, x_n]$, then we say:

- $\mathcal{C}$ **is HF-finite** if the number of numerical functions which arise as the Hilbert functions of P/I, $I \in \mathcal{C}$, is finite,
- $\mathcal{C}$ is **HP-finite** if the number of polynomials which arise as the Hilbert polynomials of P/I, $I \in \mathcal{C}$, is finite,
- $\mathcal{C}$ is **reg-limited** if for some integer t and all $I \in \mathcal{C}$ we have $reg(P/I) \leq t$,
- $\mathcal{C}$ is **g-reg-limited** if for some integer t and all $I \in \mathcal{C}$ we have $g\text{-}reg(P/I) \leq t$

  ($g\text{-}reg(P/I) = reg(P/I^{sat})$ called the geometric regularity).

Reg-limited

Fix $P = k[x_1, \ldots, x_n]$ and let $\mathcal{C}$ be a class of homogeneous ideals in P

$$\mathcal{C} \quad \text{reg-limited} \quad \Longleftrightarrow \mathcal{C} \quad \text{HF-finite}$$

- $(\Longrightarrow)$ Assume

$$t \geq reg(P/I) = reg(P/gin_{revlex}(I)) \geq m - 1$$

  where $m =$ maximum degree of the generators of $gin(I)$. Since $HF_{P/I}(n) = HF_{P/gin_{revlex}(I)}(n)$ and the monomials of degree $\leq t + 1$ in P are a finite number, the conclusion follows.

- $(\Longleftarrow)$ For the converse, since $HF_{P/I}(n) = HF_{P/Lex(I)}(n)$, if $\mathcal{C}$ is HF-finite, there are only a finite number of lexicographic ideals in P associated to $\mathcal{C}$. Then the result follows because

$$reg(P/I) \leq reg(P/Lex(I)).$$

g-reg-limited

We have seen (example) that

$$\mathcal{C} \quad \text{HP-finite} \quad \not\Longrightarrow \mathcal{C} \quad \text{reg-limited}$$

If $\mathcal{C}$ is HP-finite, then we have a uniform upper bound for the geometric regularity of P/I in $\mathcal{C}$.

$$\mathcal{C} \quad \text{HP-finite} \quad \Longrightarrow \mathcal{C} \quad \text{g-reg-limited}$$

It is a consequence of Gotzmann's result which says:

Let s be a positive integer such that

$$HP_A(X) = \binom{X+a_1}{a_1} + \binom{X+a_2-1}{a_2} + \cdots + \binom{X+a_s-(s-1)}{a_s}$$

with $a_1 \geq a_2 \geq \cdots \geq a_s \geq 0$. Then

$$reg(P/I^{sat}) \leq s-1.$$

For example, if A has dimension 1 and multiplicity e, then its Hilbert polynomial is

$$HP_A(n) = e = \binom{n}{0} + \binom{n-1}{0} + \cdots + \binom{n-(e-1)}{0}$$

so that $g-reg(R) \leq e-1$.

In particular, if A is Cohen-Macaulay of dimension 1 and multiplicity e, then $reg(A) \leq e-1$.

Kleiman's theorem: an algebraic proof

For every $d \geq 1$ we define recursively the following polynomials $F_d(X)$ with rational coefficients. We let

$$F_1(X) = X - 1, \quad F_2(X) = X^2 + X - 1$$

and if $d \geq 3$ then we let

$$F_d(X) = F_{d-1}(X) + X\binom{F_{d-1}(X) + d - 1}{d - 1}.$$

Assume $k = \overline{k}$, char $k = 0$.

Theorem

Let $A = P/I$ be a reduced equidimensional graded algebra of dimension d and multiplicity e. Then

$$reg(A) \leq F_d(e).$$

We can list the main steps of an algebraic proof (by Rossi, Trung and Valla).

Kleiman's theorem: an algebraic proof

We need k algebraically closed and of characteristic zero in order to use Bertini-type theorem on the generic hyperplane section of a reduced and non degenerate variety (see Flenner's result).

- The proof works by induction on the dimension $d \geq 2$ of $A = P/I$. We choose a generic element $z \in P_1$ and we consider

$$B := P/(I + zP)^{sat}.$$

  It is clear that $\dim(B) = d - 1$ and

$$e(A) = e(A/zA) = e(P/(I + zP)) = e(B) := e.$$

  If we assume that A is reduced and equidimensional, then B (actually a flat extension) is reduced equidimensional too (Flenner's result).

- Hence we need to relate $reg(A) =$g-$reg(A)$ in terms of $reg(B) =$ g-$reg(A/zA) \leq F_{d-1}(e)$.

Mumford's theorem: an algebraic approach

- Unlike the regularity, the geometric regularity does not behave well under generic (and regular) hyperplane sections. Take for example the standard graded algebras

$$A = k[x, y, z]/(x^2, xy), \quad T = k[x, y]/(x^2, xy).$$

Then $g\text{-}reg(A) = reg(A) = 1$ while $g\text{-}reg(T) = 0,\ reg(T) = 1.$

However the following crucial result gives us the opportunity to control this bad behaviour.

Theorem (An algebraic version of Mumford's theorem)

Let $A = P/I$ be a standard graded algebra and $z \in A_1$ a regular linear form in A. If $g\text{-}reg(A/zA) \leq m$, then

$$reg(A) \leq m + \dim(H^1(A)_m) = m + HP_A(m) - HF_A(m)$$

HF-finite

Now $HP_A(m)$ can be bounded in terms of the multiplicity and the dimension d. We can prove Kleiman's result because, by induction, we have $m = F_{d-1}(e)$, .

Corollary

Let $\mathcal{C}$ be the class of reduced equidimensional graded algebras with given multiplicity and dimension. Then $\mathcal{C}$ is HF-finite.

We need only to remark that if $P = k[x_1, \ldots, x_n]$ and $A = P/I$ has dimension d and multiplicity e, then $n - d + 1 \leq e$. The conclusion follows by Kleiman's theorem.

The theorem does not hold even if we consider reduced graded algebras not necessarily equidimensional. Take for example the graded rings

$$A_r := k[x, y, z, t, w]/(x) \cap (w, xz^r - yt^r).$$

All the elements of the family have dimension four, multiplicity one, but the regularity and the Hilbert function depends on r.

HF-finite

Kleiman's theorem does not hold if we delete the assumption that every element of the family is reduced.

Take for example the graded rings

$$A_r := k[x,y,z,t]/(y^2, xy, x^2, xz^r - yt^r).$$

This is the coordinate ring of a curve in $\mathbf{P}^3$ which can be described as the divisor $2L$ (L is a line) on a smooth surface of degree $r+1$. The Hilbert series of A_r is

$$HS_{A_r}(z) = \frac{1+2z-z^{r+1}}{(1-z)^2}$$

so that

$$\dim(A_r) = 2, \quad e(A_r) = 2$$

but $reg(A_r) = r$ and we do not have a finite number of Hilbert functions.

Contents

Bounds in terms of the degrees of generators

In the previous lectures we considered two measures of the complexity of an homogeneous ideal $I \subseteq P = k[x_1, \ldots, x_n]$:

- $d(I)$ the maximum degree of a polynomial in a minimal system of generators of I (actually of the generators of $gin_{revlex}(I)$)
- $reg(I)$: the maximum degree of the syzygies in a minimal free resolution of I

Question How much bigger can $reg(I)$ be than $d(I)$?

Obviously:

$$d(I) \leq reg(I)$$

Conjecture (Bayer '82):

$$reg(I) \leq d(I)^{2^{n-1}}$$

Bounds in terms of the degrees of generators

Giusti-Galligo ('84) : If char $k = 0$, then

$$reg(I) \leq (2d(I))^{2^{n-2}}$$

There are examples with very large regularity (Mayr-Mayer).

The regularity can really be doubly exponential in the degrees of the generators and the number of the variables.

Koh ('98) : For each integer $r \geq 1$ there exists an ideal $I_r \subseteq P = k[x_1, \ldots, x_n]$ with $n = 22r$ generated by quadrics such that

$$reg(I_r) \geq 2^{2^{r-1}}$$

These examples are highly non reduced (see also Giaimo's work for a way of making reduced examples).

Bounds in terms of the degrees of generators

Bayer-Mumford in any characteristic

$$reg(I) \leq (2d(I))^{(n-1)!}$$

In the same paper they asked whether Giusti-Galligo's bound holds in any characteristic.

Caviglia-Sbarra: If $\operatorname{ht}(I) = c < n$ and I is generated in degree $\leq d$, then

$$reg(I) \leq (d^c + (d-1)c + 1)^{2^{n-c-1}}$$

As a consequence we may deduce

- $n = 2$ $reg(I) \leq 2d$
- $n \geq 3$ $reg(I) \leq (d^2 + 2d - 1)^{2^{n-3}} \leq (2d)^{2^{n-2}}$ (Giusti-Galligo's bound) (the worst case is $\operatorname{ht}(I) = 2$.)

Bounds in terms of the degrees of generators

Problem: (Peeva-Stillman) Let $d_1 \geq d_2 \geq \dots$ be the degrees of the elements in a minimal system of generators of I. Set $c = ht(I)$, find conditions on I such that

$$\mathrm{reg}(I) \leq d_1 + \cdots + d_c - c + 1$$

Exercise.
Let $I \subseteq P = k[x_1, \dots, x_n]$, $\dim P/I = 0$, I is generated in degree $\leq d$, then

$$reg(I) \leq nd - n + 1$$

Sjögren : The previous fact holds assuming $\dim P/I \leq 1$.

For smooth (or nearly smooth) varieties there are much better bounds, linear in the degrees of the generators and in the number of the variables (see Bertram-Ein-Lazarsfeld and Chardin-Ulrich).

Eisenbud-Goto's Conjecture

Eisenbud-Goto Conjecture (84): If $\wp \subseteq (x_1, \ldots, x_n)^2$ is a prime homogeneous ideal, then

$$\mathrm{reg}(P/\wp) \leq e(P/\wp) - n + dimP/\wp$$

- It is proved for irreducible curves (Gruson, Lazarsfeld, Peskine '83)
- It is proved for smooth surfaces (Bayer-Mumford '93). Some more generality (Brodman'99)
- It is proved for some classes of toric varieties in codimension two (Peeva-Sturmfels '98)
- Slightly weaker bounds (still linear in the degree) for smooth varieties of dimension ≤ 6 (Kwak 2000)

Regularity of the Tangent Cone

Let $A = k[[x_1, \ldots, x_n]]/I$ a local ring and let m be its maximal ideal.
We define the homogeneous k-standard algebra

$$gr_m(A) = \oplus_{n \geq 0} m^n/m^{n+1}$$

which is called the associated graded ring or the tangent cone of A.

Geometric meaning: If A is the localization at the origin of the coordinate ring of an affine variety V passing through 0, then $gr_m(A)$ is the coordinate ring of the *tangent cone* of V, which is the cone composed of all lines that are limiting positions of secant lines to V in 0.

We have the following presentation

$$gr_m(A) \simeq k[x_1, \ldots, x_n]/I^*$$

where I^* is the ideal generated by the initial forms (w.r.t. the m-adic filtration) of the elements of I. The ideal I^* can by computed by using a slight modification of Buchberger's algorithm (see SINGULAR).

Example

Example

Consider the power series $A = k[[t^4, t^5, t^{11}]]$. This is a one-dimensional local domain and

$$A = k[[x, y, z]]/I \quad \text{where} \quad I = (x^4 - yz, y^3 - xz, z^2 - x^3y^2).$$

We can prove that

$$gr_m(A) = k[x, y, z]/(xz, yz, z^2, y^4)$$

We have $\dim A = \dim\ gr_m(A) = 1$, but depth $gr_m(A) = 0$.

We always have $\dim A = \dim\ gr_m(A)$, but the above example shows that

$$A \ \text{Cohen-Macaulay} \not\Longrightarrow gr_m(A) \ \text{Cohen-Macaulay}$$

Minimal free resolution of the tangent cone

Denote by $\mu(\ \ )$ the minimal number of generators of an ideal of A. The Hilbert function of A is, by definition

$$HF_A(n) := dim_k\ m^n/m^{n+1} = \mu(m^n)$$

for every $n \geq 0$. Hence HF_A is the Hilbert function of the homogeneous k-standard algebra

$$gr_m(A) = \oplus_{n\geq 0} m^n/m^{n+1}$$

In particular $e(A) = e(gr_m(A)),\ \dim A = \dim gr_m(A).$ Several papers have

been produced concerning the following problem:

Problem: Compare the numerical invariants of the R-free minimal resolution of A ($R = k[[x_1, \ldots, x_n]]$) with those of the P-free minimal graded resolution ($P = k[x_1, \ldots, x_n]$) of $gr_m(A)$:

$$0 \to R^{\beta_h(I)} \to R^{\beta_{h-1}(I)} \to \cdots \to R^{\beta_0(I)} \to I \to 0$$

$$0 \to P^{\beta_s(I^*)} \to P^{\beta_{s-1}(I^*)} \to \cdots \to P^{\beta_0(I^*)} \to I^* \to 0$$

Minimal free resolution of the tangent cone

$$\beta_i(I) \leq \beta_i(I^*)$$

In general is $<$ (see R.-Sharifan for more complete information).

Example (Herzog,R., Valla)

Consider $I = (x^3 - y^7, x^2y - xt^3 - z^6)$ in $R = k[[x, y, z, t]]$. Since I is a complete intersection, then a minimal free resolution of I is given by:

$$0 \to R \to R^2 \to I \to 0.$$

But

$$I^* = (x^3, x^2y, x^2t^3, xt^6, x^2z^6, xy^9 - xz^6t^3, xy^8t^3, y^7t^9),$$

hence $\mu(I^*) = 8$ and a minimal free resolution of I^* is given by

$$0 \to P \to P^6 \to P^{12} \to P^8 \to I^* \to 0$$

In particular depth $A = 2$ and depth $gr_m(A) = 0$.

Regularity of $gr_m(A)$

It is an interesting problem to study the Castelnuovo-Mumford regularity of the tangent cone of a Cohen-Macaulay local ring.

- If $gr_m(A)$ is a Cohen-Macaulay graded algebra, then

$$reg(gr_m(A)) \leq e(A) - n + d$$

- A 1-dimensional Cohen-Macaulay then

$$reg(gr_m(A)) \leq e(A) - 1.$$

Problem. [R., Trung, Valla] Let (A, m) be a local Cohen-Macaulay ring. Is $reg(gr_m(A))$ bounded by a polynomial function (possibly linear) of the multiplicity $e(A)$ and the codimension?

Srinivas-Trivedi, Rossi-Trung-Valla proved very large bounds.

Regularity of $G = gr_m(A)$

The following results allow to repeat the procedure of Lesson 4 (Mumford's inequality) for studying $reg(G)$.

- Assume that $depth A > 0$. Then

$$reg(G) = \text{g-}reg(G).$$

- Let x be a generic element of $m - m^2$ and $\overline{G} = gr_{m/(x)}(A/(x))$. Then

$$\text{g-}reg(G/(x^*)) = \text{g-}reg(\overline{G}).$$

Theorem (R, Valla, Trung)

Let A be a Cohen-Macaulay local ring with $d = \dim A \geq 1$. Then

(i) $\operatorname{reg}(G) \leq e(A) - 1$ *if* $d = 1$,

(ii) $\operatorname{reg}(G) \leq e(A)^{2((d-1)!)-1}[e(A) - 1]^{(d-1)!}$ *if* $d \geq 2$.

Finiteness of HF

As in Kleiman's theorem (Lesson 4), as an application of the bound on the Castelnuovo-Mumford regularity, we obtain the finiteness of Hilbert functions of local rings with given dimension and multiplicity.

Theorem (Srinivas, Trivedi; R, Valla, Trung)

Given two positive integers d and q there exist only a finite number of Hilbert functions for a local Cohen-Macaulay ring A with $\dim A = d$ and $e \leq q$.

Local version of Kleiman's Theorem?

We remark that the analogous of Kleiman result does not hold in the local case.

Srinivas and Trivedi showed with the following example that the class of local domains of dimension two and multiplicity 4 does not have a finite number of Hilbert functions. Let

$$A_r := k[[X, Y, Z, T]]/\wp_r$$

where

$$\wp_r = (Z^r T^r - XY, X^3 - Z^{2r} Y, Y^3 - T^{2r} X, X^2 T^r - Y^2 Z^r).$$

Then it is easy to see that $\wp_r$ is a prime ideal and the associated graded ring of A_r is the standard graded algebra

$$G_r = k[X, Y, Z, T]/(XY, X^3, Y^3, X^2 T^r - Y^2 Z^r).$$

We have

$$\mathrm{reg}(G_r) = r + 1, \quad HS_{A_r}(z) = \frac{1 + 2z + 2z^2 - z^{r+2}}{(1-z)^2}.$$

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