Local cohomology modules and derived functors

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General notations and terminology

1. $R$: Commutative Noetherian ring with non-zero identity
2. $\mathfrak{a}$: An ideal of $R$
3. $M$: An $R$-module
4. $\mathbb{N}_0$ (resp. $\mathbb{N}$): The set of non-negative (resp.) positive integers.
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Recall that:

For each $R$-module $M$, set 
$$\Gamma_a(M) := \bigcup_{n \in \mathbb{N}} (0 : M \ a^n)$$

Also for a homomorphism $f : M \to N$ of $R$-modules, we set $\Gamma_a(f)$ is the restriction of $f$ to $\Gamma_a(M)$. Note that $f(\Gamma_a(M)) \subseteq \Gamma_a(N)$. Thus $\Gamma_a(-)$ becomes a covariant, $R$-linear, left exact functor from the category of $R$-modules and $R$-homomorphisms to itself. We call $\Gamma_a(-)$ the $a$-torsion functor. For $i \geq 0$, the $i$-th right derived functor of $\Gamma_a(-)$ is denoted by $H^i_a(-)$ and will be referred to as the $i$-th local cohomology functor with respect to $a$. 
Definition:

There is a canonical map

$$\mu_M : R \longrightarrow \text{End}_R(M)$$

such that for $r \in R$, $\mu_M(r)$ is the multiplication map by $r$ on $M$.

It is easy to see that $\mu_M$ is a homomorphism of $R$-algebras. In general, $\mu_M$ is neither injective nor surjective.

Let $(R, m)$ be a Noetherian local ring. Let $D(-)$ be the Matlis dual functor $\text{Hom}_R(-, E)$, where $E$ is the injective hull of the field $R/m$. 
Let \( R \) be a local ring. \( M \) has a canonical embedding

\[
M \longrightarrow D(D(M)) = D^2(M),
\]

\[
m \longmapsto (\varphi \longmapsto \varphi(m))
\]

into its bidual, this map will denoted by \( i_M \). We will consider \( M \) as a submodule of \( D^2(M) \) via \( i_M \).
Definition. For an $R$-module $M$, the cohomological dimension of $M$ with respect to $\alpha$ is defined as

$$\text{cd}(\alpha, M) := \max\{i \in \mathbb{Z} \mid H^i_\alpha(M) \neq 0\}.$$
Introduction

Let \((R, \mathfrak{m})\) be a Noetherian local ring.

For a positive integer \(n\), by using the theory of D-modules, Hellus showed that \(H^n_{\mathfrak{a}}(D(H^n_{\mathfrak{a}}(R)))\) is either \(E\) or zero in the following cases:

\((\alpha)\) \(R\) is a Noetherian local complete Cohen-Macaulay ring with coefficient field \(R/\mathfrak{m}\) and there exists a regular sequence \(x_1, \ldots, x_n \in \mathfrak{a}\) on \(R\) such that \(\mathfrak{a} = (x_1, \ldots, x_n)\). In this case \(\mathfrak{a}\) is a set-theoretic complete intersection ideal of \(R\).

\((\beta)\) \(R\) is a Noetherian local complete regular ring of equicharacteristic zero and a an ideal of height \(n \geq 1\) such that there exists a regular sequence \(x_1, \ldots, x_n \in \mathfrak{a}\) on \(R\) and \(H^i_{\mathfrak{a}}(R) = 0\) for every \(i > n\).
In [*], the present author obtained the following generalization of Hellus’ Theorem.

**Theorem** Let $R$ be a local ring and $a$ be an ideal of $R$ such that $aM \neq M$ and $n := \text{grade}_M a = \text{cd}(a, M) \geq 1$. Then

$$H^n_a(D(H^n_a(M))) \cong D(M).$$

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**Theorem** Let $R$ be a local ring and $\alpha$ be an ideal of $R$ such that $\alpha M \neq M$ and $n := \text{grade}_M \alpha = \text{cd}(\alpha, M) \geq 1$. Then

$$H^n_{\alpha}(D(H^n_{\alpha}(M))) \cong D(M).$$

By using this generalization in conjunction with spectral sequences method, Hellus and Stückrad, in [*], showed that:

if $R$ is Noetherian local complete and $\mathfrak{a}$ an ideal of $R$ such that $H^i_\mathfrak{a}(R) = 0$ for every $i \neq n (= \text{height } \mathfrak{a})$, then $\mu_{H^n_\mathfrak{a}(R)}$ is bijective.

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Moreover, Hellus and Stückrad, raised the following question:

If $R$ is a commutative Noetherian complete local ring and $\underline{x} := x_1, \ldots, x_n$ is a regular sequence on $R$ contained in $\mathfrak{a}$, when exactly is $J_{\underline{x}, \mathfrak{a}, R} := D(H^n_{\underline{x}R}(D(H^n_{\mathfrak{a}}(R))))$ zero?

where $\underline{x}R$ is the ideal $\sum_{i=1}^n x_i R$ of $R$. 
Moreover, Hellus and Stückrad, raised the following question:

If $R$ is a commutative Noetherian complete local ring and $\overline{x} := x_1, \ldots, x_n$ is a regular sequence on $R$ contained in $a$, when exactly is $J_{\overline{x}, a, R} := D(H^n_{\overline{x}R}(D(H^n_a(R))))$ zero?

where $\overline{x}R$ is the ideal $\sum_{i=1}^n x_i R$ of $R$. 
Let \((R, m)\) be a Noetherian local ring and \(x \coloneqq x_1, \ldots, x_h\) a sequence of \(R\). For every \(R\)-module \(M\) there is a canonical map

\[
M/xM \xrightarrow{i_{M,x}} H^h_{xR}(M)
\]

(coming from the description

\[
H^h_{xR}(M) \cong \lim_{n \in \mathbb{N}} M/(x^n_1, \ldots, x^n_h)M.
\]
Theorem: Let \((R, m)\) be a Noetherian local complete ring and \(a\) an ideal of \(R\) such that \(H^\ell_a(R) = 0\) for every \(\ell \neq h = \text{height}(a)\). Set \(H := H^h_a(R)\). Then

1. \(\text{Hom}(H, i_H) : \text{End}(H) \longrightarrow \text{Hom}(H, D^2(H))\) is an isomorphism.

2. There is a canonical isomorphism

\[ \gamma_H : \text{Hom}(H, D^2(H)) \longrightarrow D(H^h_a(D(H))). \]

3. \(\mu_H : R \longrightarrow \text{End}(H)\) is an isomorphism of \(R\)-algebras.

4. Consequently there is a canonical isomorphism

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Theorem: Let \((R, m)\) be a Noetherian local complete ring and \(a\) an ideal of \(R\) such that \(H^\ell_a(R) = 0\) for every \(\ell \neq h = \text{height}(a) \geq 1\); let \(x := x_1, \ldots, x_h \in a\) be an \(R\)-regular sequence. Set \(D := D(H^h_a(R))\).

The following conditions are equivalent:

1. \(\sqrt{a} = \sqrt{(xR)}\).
2. \(x\) is a sequence on \(D\).
3. \(D/\langle x \rangle D \xrightarrow{i_{D,x}} H^h_{xR}(D)\) is injective.
4. \(J_{x,a} = R\).
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4. \(J^h_{x,a} = R\).
**Question.** Let \((R, m)\) be a Noetherian local complete ring and \(\mathfrak{a}\) an ideal of \(R\), \(h \in \mathbb{N}\); assume that \(x := x_1, \ldots, x_h \in \mathfrak{a}\) is an \(R\)-regular sequence. When

\[ J_{x, \mathfrak{a}, R} := 0. \]
Theorem. Let \((R, m)\) be a Noetherian local Gorenstien ring of dimension \(n\) and \(a\) an ideal of \(R\) such that \(\dim R/a = n - c\). Then there is a natural isomorphism

\[
\text{End}_R(H^c_a(R)) \cong \text{Ext}^c_R(H^c_a(R), R)
\]
Recall that we say a sequence of elements $x_1, \ldots, x_k$ of $a$ is an $a$-filter regular sequence on $M$ if

$$x_i \notin \bigcup_{p \in \text{Ass}_R \left( \frac{M}{(x_1, \ldots, x_{i-1})M} \right) \setminus V(a)} p$$

for $i = 1, \ldots, k$. 

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Lemma. Let $R$ is Noetherian, $M$ is a finitely generated. If $x_1, \ldots, x_n$ be an $\alpha$-filter regular sequence on $M$, then there is an element $x_{n+1} \in \mathfrak{a}$ such that $x_1, \ldots, x_n, x_{n+1}$ is an $\alpha$-filter regular sequence on $M$.

Lemma. Let $n > 1$ and $x_1, \ldots, x_n$ be an $\alpha$-filter regular sequence on $M$. Then

$$H^i_\alpha(M) \cong \begin{cases} H^i_{(x_1, \ldots, x_n)}(M) & \text{for } 0 \leq i < n, \\ H^{i-n}_\alpha(H^n_{(x_1, \ldots, x_n)}(M)) & \text{for } n \leq i. \end{cases}$$
Lemma. Let $R$ is Noetherian, $M$ is a finitely generated. If $x_1, \ldots, x_n$ be an $a$-filter regular sequence on $M$, then there is an element $x_{n+1} \in a$ such that $x_1, \ldots, x_n, x_{n+1}$ is an $a$-filter regular sequence on $M$.

Lemma. Let $n > 1$ and $x_1, \ldots, x_n$ be an $a$-filter regular sequence on $M$. Then

$$H^i_a(M) \cong \begin{cases} H^i_{(x_1,\ldots,x_n)}(M) & \text{for } 0 \leq i < n, \\ H^i_{a^{-n}}(H^n_{(x_1,\ldots,x_n)}(M)) & \text{for } n \leq i. \end{cases}$$
Proposition. For a non-negative integer $n$ and an $\alpha$-filter regular sequence $x_1, \ldots, x_{n+1} \in \alpha$ on $M$, there exists an exact sequence

\[ 0 \longrightarrow H^n_\alpha(M) \longrightarrow H^n_{(x_1, \ldots, x_n)}(M) \longrightarrow (H^n_{(x_1, \ldots, x_n)}(M))_{x_{n+1}} \]
\[ \longrightarrow H^{n+1}_{(x_1, \ldots, x_{n+1})}(M) \longrightarrow 0. \]
Proposition. Let $n$ be a non-negative integer and $x_1, \ldots, x_n$ be an $\alpha$-filter regular sequence on $M$. Let $T$ be an $\alpha$-torsion $R$-module. Then

$$\text{Hom}_R(T, H^n_\alpha(M)) \cong \text{Hom}_R(T, H^n_{(x_1, \ldots, x_n)}(M)).$$

In particular

$$\text{End}_R(H^n_\alpha(M)) \cong \text{Hom}_R(H^n_\alpha(M), H^n_{(x_1, \ldots, x_n)}(M)).$$
**Theorem.** Let $\mathfrak{a}$ be a proper ideal of $R$ and $n := \text{grade}_R \mathfrak{a}$. Then, for every $\mathfrak{a}$-torsion $R$-module $T$, we have the following isomorphism

$$\text{Hom}_R(T, H^n_\mathfrak{a}(R)) \cong \text{Ext}_R^n(T, R).$$

In particular

$$\text{End}_R(H^n_\mathfrak{a}(R)) \cong \text{Ext}_R^n(H^n_\mathfrak{a}(R), R).$$
**Theorem:** Let \( \mathfrak{a} \) be a proper ideal of \( R \) such that \( n := \text{grade}_R \mathfrak{a} = \text{cd}(\mathfrak{a}, R) \). Let \( \text{Ext}^i_R(R_z, R) = 0 \) for all \( i \in \mathbb{N} \) and \( z \in \mathfrak{a} \). Then

1. \( \text{End}_R(H^n_\mathfrak{a}(R)) \) is a homomorphic image of \( R \).
2. If moreover \( \text{Hom}_R(R_z, R) = 0 \) for all \( z \in \mathfrak{a} \), then \( \text{End}_R(H^n_\mathfrak{a}(R)) \cong R \) and so \( \mu_{H^n_\mathfrak{a}(R)} \) is bijective.
Theorem: Let $\mathfrak{a}$ be a proper ideal of $R$ such that $n := \text{grade}_{R}\mathfrak{a} = \text{cd}(\mathfrak{a}, R)$. Let $\text{Ext}^i_R(R_z, R) = 0$ for all $i \in \mathbb{N}$ and $z \in \mathfrak{a}$. Then

1. $\text{End}_R(H^n_{\mathfrak{a}}(R))$ is a homomorphic image of $R$.
2. If moreover $\text{Hom}_R(R_z, R) = 0$ for all $z \in \mathfrak{a}$, then $\text{End}_R(H^n_{\mathfrak{a}}(R)) \cong R$ and so $\mu_{H^n_{\mathfrak{a}}(R)}$ is bijective.
Theorem: Let \( a \) be a proper ideal of \( R \) such that
\[
n := \text{grade}_{R} a = \text{cd}(a, R).
\]
Let \( \text{Ext}^i_{R}(R_z, R) = 0 \) for all \( i \in \mathbb{N} \) and \( z \in a \). Then

1. \( \text{End}_R(H^n_a(R)) \) is a homomorphic image of \( R \).
2. If moreover \( \text{Hom}_R(R_z, R) = 0 \) for all \( z \in a \), then
   \( \text{End}_R(H^n_a(R)) \cong R \) and so \( \mu_{H^n_a(R)} \) is bijective.
Corollary. Let \((\mathcal{R}, \mathcal{m})\) be a Noetherian local complete ring and \(\mathcal{a}\) an ideal of \(\mathcal{R}\) such that \(n \equiv \text{grade}\mathcal{R}\mathcal{a} = \text{cd}(\mathcal{a}, \mathcal{R})\). Set \(\mathcal{H} \equiv \mathcal{H}_\mathcal{a}^n(\mathcal{R})\). Then

\[
\mu_\mathcal{H} : \mathcal{R} \longrightarrow \text{End}_\mathcal{R}(\mathcal{H})
\]

is an isomorphism of \(\mathcal{R}\)-algebras.
Theorem. Let $F$ be an $R$-linear covariant functor from $\mathcal{C}(R)$ to itself such that for every $R$-module $L$, $F(L)$ is $\alpha$-torsion. Also let $c \in \mathbb{N}_0$ and $\alpha$ be an ideal of $R$ such that $\alpha M \neq M$ and that $c \leq \text{grade}(\alpha, M)$. Then

$$\mathcal{R}^0 F(H^c_\alpha(M)) \cong \mathcal{R}^c F(M).$$
**Theorem.** Let $F$ be an $R$-linear covariant functor from $\mathcal{C}(R)$ to itself such that for every $R$-module $L$, $F(L)$ is $\alpha$-torsion. Suppose that $\alpha$ is an ideal of $R$ and $M$ is a finitely generated $R$-module such that $\alpha M \neq M$ and that $c := \text{cd}(\alpha, M) = \text{grade}(\alpha, M)$. Then

$$\mathcal{R}^i F(H^c_\alpha(M)) \cong \mathcal{R}^{i+c} F(M)$$

for all $i \in \mathbb{N}_0$. 
Theorem. Let $M$ be a finitely generated $R$-module, $a$ be an ideal of $R$ such that $aM \neq M$ and $c := \text{cd}(a, M) = \text{grade}(a, M)$. Then, for every ideal $b$ of $R$ with $b \supseteq a$, 

(i) $H^i_b(H^c_a(M)) \cong H^{i+c}_b(M)$, and;

(ii) $\text{Ext}^i_R(R/b, H^c_a(M)) \cong \text{Ext}^{i+c}_R(R/b, M)$

for all $i \in \mathbb{N}_0$. 
**Theorem.** Let \((R, m)\) be a Gorenstein local ring and \(a\) be a cohomological complete intersection ideal of \(R\). Set \(c := \text{cd}(a, R)\) and \(d := \dim_R R/a\). Then

(i) \(H^d_m(H^c_a(R)) \cong E(R/m)\),

(ii) \(\text{Ext}^d_R(R/m, H^c_a(R)) \cong E(R/m)\), and;

(iii) \(H^i_m(H^c_a(R)) = 0 = \text{Ext}^i_R(R/m, H^c_a(R))\) for all \(i \neq d\).
Theorem. Let $a$ and $b$ be ideals of an arbitrary commutative Noetherian ring $R$ such that $b \supseteq a$, $aM \neq M$ and $c := \text{grade}(a, M)$. Then

(i) we have a monomorphism from $H^c_b(M)$ to $H^c_a(M)$, and;
(ii) there exists a natural homomorphism from $\text{End}(H^c_a(M))$ to $\text{End}(H^c_b(M))$. 
Module of generalized fractions

Let $M$ be an $R$-module. The construction of a module of generalized fractions of $M$ requires a (positive integer $n$ and a) triangular subset $U \subseteq R^n$; the construction produces a module $U^{-n}M$, called the module of generalized fractions of $M$ with respect to $U$, whose elements, called generalized fractions, have the form $\frac{m}{(u_1, \ldots, u_n)}$, where $m \in M$ and $(u_1, \ldots, u_n) \in U$.

The concept of a chain of triangular subsets on $R$ is explained in [*]. Such a chain $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ determines a complex of modules of generalized fractions

$$0 \xrightarrow{d^{-1}} M \xrightarrow{d^0} U_1^{-1}M \xrightarrow{} \ldots \xrightarrow{} U_i^{-1}M \xrightarrow{d^i} U_{i+1}^{-1}M \xrightarrow{} \ldots,$$

in which $d^0(m) = m/(1)$ for all $m \in M$ and $d^i(m/(u_1, \ldots, u_i)) = m/(u_1, \ldots, u_i, 1)$ for all $i \in \mathbb{N}$, $m \in M$ and $(u_1, \ldots, u_i) \in U_i$. We shall denote this complex by $C(\mathcal{U}, M)$.

Let $\bar{x} := x_1, \ldots, x_n$ be a sequence of elements of $R$. For each $i \in \mathbb{N}$, set

$$U(x)_i := \{(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i}) : \text{there exists } j \text{ with } 0 \leq j \leq i \text{ such that }$$

$$\alpha_1, \ldots, \alpha_j \in \mathbb{N} \text{ and } \alpha_{j+1} = \cdots = \alpha_i = 0\},$$

where $x_r$ is interpreted as 1 whenever $r > n$. It is easy to see that, for each $i \in \mathbb{N}$, $U(x)_i$ is a triangular subset of $R^i$. We use $R(x)$ to denote the family $(U(x)_i)_{i \in \mathbb{N}}$. Hence $R(x)$ is a chain of triangular subsets on $R$. Write the associated complex $C(R(x), M)$ as

$$0 \xrightarrow{d_{X,M}^{-1}} M \xrightarrow{d_{X,M}^0} U(x)_1^{-1} M \xrightarrow{d_{X,M}^i} \cdots \xrightarrow{d_{X,M}^i} U(x)_{i+1}^{-1} M \xrightarrow{d_{X,M}^i} \cdots$$
Proposition Let $\mathfrak{a}$ be a proper ideal of a Noetherian local ring $R$. Let $x := x_1, \ldots, x_n (n > 0)$ be a regular sequence on $M$ contained in $\mathfrak{a}$. Then there exists an exact sequence

$$0 \longrightarrow J_{x, \mathfrak{a}, M} \longrightarrow D(D(M)) \longrightarrow D(H^{n-1}_{x, R}(D(Ker d^n_{y, M})))$$

for every $x_{n+1} \in \mathfrak{a}$ such that $y := x_1, \ldots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on $M$. 
Theorem Let \((R, m)\) be a Noetherian local ring and \(\mathfrak{a}\) be a proper ideal of \(R\). Let \(x := x_1, \ldots, x_n (n > 0)\) be a regular sequence on \(M\) in \(\mathfrak{a}\). Suppose that there exists \(x_{n+1} \in \mathfrak{a}\) such that \(y := x_1, \ldots, x_n, x_{n+1}\) is an \(\mathfrak{a}\)-filter regular sequence on \(M\) and \(H^n_{\mathfrak{a}R}(D(U(y)_{n+1}^{-1} M)) = 0\). Then

\[ J_{\mathfrak{a}, M} \cong D(D(M)). \]
Thanks For Your Patience