Boij-Söderberg theory

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Conjectures

**Conjecture 1**

*All extremal rays of pure type do exist:* For any sequence of integers \(d_0 < d_1 < \cdots < d_c\) there exists a CM-module of codimension \(c\) with resolution

\[
S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_c)^{\beta_c}.
\]

**Conjecture 2**

Pure diagrams account for *all* the extremal rays: There are no more extremal rays in the cone of Betti diagrams than those coming from pure diagrams.
Conjecture 3

The algorithm always works in order to write a Betti diagram $\beta$ as a positive linear combination of pure diagrams: It gives a chain of degree sequences $d^1 < d^2 < \cdots < d^r$ such that

$$\beta = c_1 \pi(d^1) + c_2 \pi(d^2) + \cdots + c_r \pi(d^r).$$
Conjectures

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Note. Conjecture 3 implies conjecture 2.
Koszul complexes

Example

Pure resolution of type \((0,1,2,3)\).

\[ S \leftarrow S(-1)^3 \leftarrow S(-2)^3 \leftarrow S(-3). \]
Koszul complexes

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Let \( S = Symm(V) \) where \( V \) is a \( k \)-vector space of dimension \( n \). Consider \( \wedge^p V \) to have degree \( p \). Pure resolution of type \((0, 1, 2, \ldots, n)\):

\[ S \leftarrow S \otimes_k V \leftarrow S \otimes_k \wedge^2 V \leftarrow \cdots \leftarrow S \otimes_k \wedge^n V. \]
Powers of maximal ideals

Example

Let \( m \subseteq k[x_1, x_2, x_3] \) be the maximal ideal. Resolution of \( m^2 \) has type \((0, 2, 3, 4)\):

\[
S \leftarrow S(-2)^6 \leftarrow S(-3)^8 \leftarrow S(-4)^3.
\]

Let \( S_r(V) \) be the \( r \)'th graded piece of \( S = Symm(V) \). Let \( m \) the maximal ideal in \( S \). The resolution of \( m^r \) has type \((0, r, r + 1, r + 2, \ldots, n + r - 1)\):

\[
S \leftarrow S \otimes_k S_r(V) \leftarrow S \otimes_k S_{r,1}(V) \leftarrow \cdots \leftarrow S \otimes_k S_{r,1,\ldots,1}(V) \leftarrow \cdots.
\]
The general linear group $GL(V)$ consists of all automorphisms of the vector space $V$. Let $n = \dim_k V$. For every partition into $n$ parts

$$\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

there is an irreducible $GL(V)$-representation, denoted $S_\lambda(V)$.

**Example**

$$S_{1,1,\ldots,1}(V) = \wedge^r V, \quad S_{r,0,\ldots,0}(V) = S_r(V).$$

$r$ copies of 1
Pure resolutions of type length three

char. $k = 0$ (Eisenbud, F., Weyman)

Let $S = k[x_1, x_2, x_3]$. Want pure resolution:

$$S^\beta_0 \leftarrow S(-e_1)^\beta_1 \leftarrow S(-e_1 - e_2)^\beta_2 \leftarrow S(-e_1 - e_2 - e_3)^\beta_3.$$
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Try:

$$S \otimes_k S_{\lambda_1, \lambda_2, \lambda_3} \leftarrow S \otimes_k S_{\lambda_1 + e_1, \lambda_2, \lambda_3} \leftarrow S \otimes_k S_{\lambda_1 + e_1, \lambda_2 + e_2, \lambda_3} \leftarrow S \otimes_k S_{\lambda_1 + e_1, \lambda_2 + e_2, \lambda_3 + e_3}.$$
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This works! Let:

$$\lambda_3 = 0, \quad \lambda_2 = e_3 - 1, \quad \lambda_1 = (e_2 - 1) + (e_3 - 1).$$
Pure resolutions of type length three

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The construction generalizes to resolutions of any length \( n \).
Divided powers $D_r(V) = S_r(V^*)^*$. Let

$$\tilde{D}_r(V) = D_r(V) \otimes \wedge^\dim_k V V.$$
Eagon-Northcott complex

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$$\tilde{D}_r(V) = D_r(V) \otimes \wedge^{\dim_k V} V.$$ 

Let $A$ and $B$ be vector spaces with $b = \dim_k B \leq \dim_k A$.

$$S \leftarrow \wedge^b A \otimes \tilde{D}_0(B^*) \otimes S(-b) \leftarrow \wedge^{b+1} A \otimes \tilde{D}_1(B^*) \otimes S(-b-1) \leftarrow \wedge^{b+2} A \otimes \tilde{D}_2(B^*) \otimes S(-b-2) \leftarrow \cdots$$
Extends to family of complexes
Pure resolutions with two linear parts

Buchsbaum-Rim complex:

\[ S_1(B) \otimes S \leftarrow A \otimes S_0(B) \otimes S(-1) \leftarrow \wedge^{b+1} A \otimes \tilde{D}_0(B^*) \otimes S(-b - 1) \leftarrow \wedge^{b+2} A \otimes \tilde{D}_1(B^*) \otimes S(-b - 2) \]
Boij-Söderberg conjectures
Construction of pure resolutions
The equations of the exterior facets of the cone
Pairings with cohomology tables of vector bundles on projectives

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Further complexes:

\[ S_2(B) \otimes S \leftarrow A \otimes S_1(B) \otimes S(-1) \leftarrow \wedge^2 A \otimes S_0(B) \otimes S(-2) \]
\[ \leftarrow \wedge^{b+2} A \otimes \tilde{D}_0(B^*) \otimes S(-b-2) \leftarrow \wedge^{b+3} A \otimes \tilde{D}_1(B^*) \otimes S(-b-3) \]
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\[ S_p(B) \otimes S \leftarrow \cdots \leftarrow \wedge^p A \otimes S_0(B) \otimes S(-p) \]
\[ \leftarrow \wedge^{p+b} A \otimes \tilde{D}_0(B^*) \otimes S(-p - b) \leftarrow \wedge^{p+b+1} A \otimes \tilde{D}_1(B^*) \otimes S(-p - b) \]
Two vector spaces $B_1$ and $B_2$ of dimensions $b_1$ and $b_2$.

\[ S_p(B_1) \otimes S_q(B_2) \otimes S \leftarrow A \otimes S_{p-1}(B_1) \otimes S_{q-1}(B_2) \otimes S(-1) \leftarrow \cdots \]

\[ \leftarrow \wedge^p A \otimes S_0(B_1) \otimes S_{q-p}(B_2) \otimes S(-p) \]

twist jump
Pure resolutions with three linear parts

(Eisenbud, Schreyer), (Berkesch, Erman, Kummini, Sam)

Two vector spaces $B_1$ and $B_2$ of dimensions $b_1$ and $b_2$.

$S_p(B_1) \otimes S_q(B_2) \otimes S \leftarrow A \otimes S_{p-1}(B_1) \otimes S_{q-1}(B_2) \otimes S(-1) \leftarrow \ldots$

$\leftarrow \wedge^p A \otimes S_0(B_1) \otimes S_{q-p}(B_2) \otimes S(-p)$

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$$\leftarrow \wedge^p A \otimes S_0(B_1) \otimes S_{q-p}(B_2) \otimes S(-p)$$

**twist jump**

$$\leftarrow \wedge^{p+b_1} A \otimes \tilde{D}_0(B_1^*) \otimes S_{q-p-b_1}(B_2) \otimes S(-p - b_1)$$

$$\leftarrow \wedge^{p+b_1+1} A \otimes \tilde{D}_1(B_1^*) \otimes S_{q-p-b_1-1}(B_2) \otimes S(-p - b_1 - 1) \leftarrow \cdots$$

$$\leftarrow \wedge^q A \otimes \tilde{D}_{q-p-b_1}(B_1^*) \otimes S_0(B_2) \otimes S(-q)$$

**twist jump**

$$\leftarrow \wedge^{q+b_2} \otimes \tilde{D}_{q-p-b_1-b_2}(B_1^*) \otimes \tilde{D}_0(B_2^*) \otimes S(-q - b_2) \cdots$$

$$\leftarrow \wedge^{q+b_2+1} \otimes \tilde{D}_{q-p-b_1+b_2+1}(B_1^*) \otimes \tilde{D}_1(B_2^*) \otimes S(-q - b_2 - 1) \cdots$$
The simplicial fan

Fix two degree sequences
\[ a = (a_0 < a_1 < \ldots < a_n), \quad b = (b_0 < b_1 < \ldots < b_n). \]

Consider chain of degree sequences
\[ a \leq d^1 < d^2 < \cdots < d^r \leq b. \]
The simplicial fan

Fix two degree sequences

\[ a = (a_0 < a_1 < \ldots < a_n), \quad b = (b_0 < b_1 < \ldots < b_n). \]

Consider chain of degree sequences

\[ a \leq d^1 < d^2 < \ldots < d^r \leq b. \]

Get pure diagrams

\[ \pi(d^1), \pi(d^2), \ldots, \pi(d^r). \]

These are linearly independent so they generate a simplicial cone. Varying over all chains, we get a simplicial fan \( F \).
Geometric version of Boij-Söderberg conjectures

BS conjecture 1 says: $|F| \subseteq B$.

BS conjecture 3 says: $B \subseteq |F|$.

Cone $B$ of Betti diagrams

Simplicial fan $F$ generated by pure diagrams
Geometric version of Boij-Söderberg conjectures

BS conjecture 1 says:
\[ |F| \subseteq B. \]

BS conjecture 3 says:
\[ B \subseteq |F|. \]

Conclusion: \[ B = |F|. \]

Note that BS conjecture 3 implies BS conjecture 2.

Cone \( B \) of Betti diagrams

Simplicial fan \( F \) generated by pure diagrams
Strategy for proof
(Eisenbud, Schreyer)

BS conjecture 1: The existence of pure resolutions shows that $|F| \subseteq B$. 
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- Find equations $h$ of supporting hyperplanes for the exterior facets of $F$. 
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BS conjecture 3:
- Find equations $h$ of supporting hyperplanes for the exterior facets of $F$.
- For each exterior facet equation $h$ and Betti diagram $\beta$ of a Cohen-Macaulay module, show that $h(\beta) \geq 0$. 
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BS conjecture 1: The existence of pure resolutions shows that $|F| \subseteq B$.

BS conjecture 3:
- Find equations $h$ of supporting hyperplanes for the exterior facets of $F$.
- For each exterior facet equation $h$ and Betti diagram $\beta$ of a Cohen-Macaulay module, show that $h(\beta) \geq 0$.
- This shows that $B \subseteq |F|$.
Facet equations

Exterior facets have equations:

$$
\sum_{i=0, \ldots, c} c_{ij} \beta_{ij} = 0,
$$

which we represent by an array:

\[\begin{array}{c|ccc}
\text{i} & \text{j} & \cdots & c_{ij} \\
\end{array}\]
Facet equations

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$$\sum_{i=0,\ldots,c} c_{ij} \beta_{ij} = 0,$$

which we represent by an array:

$$\begin{array}{c|c|c|c}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdot & \cdot & \cdot & \cdot \\
i & & & j \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
j & \cdots & c_{ij} & \\
\end{array}$$

There are three types of exterior facets. Types 1 and 2 have the simple equations $\beta_{ij} = 0$ for suitable $i$ and $j$. 
Facet equations of type 3

Example

There is a maximal chain

\[ \mathbf{a} = (0, 1, 3) < (0, 2, 3) < (0, 2, 4) < (0, 3, 4) = \mathbf{b}. \]

Get four-dimensional simplicial cone generated by

\[ \pi(0, 1, 3) = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \pi(0, 2, 3), \pi(0, 2, 4), \pi(0, 3, 4). \]
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There is a maximal chain

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The facet generated by

\[ \pi(0, 1, 3), \pi(0, 2, 3), \pi(0, 3, 4) \]

is an exterior facet of type 3.
Facet equations of type 3

Exterior facets of type 3 occurs when the maximal chain is

\[ \cdots < (\ldots, r-1, r, \ldots) < (\ldots, r-1, r+1, \ldots) < (\ldots, r, r+1, \ldots) < \cdots, \]

and we form the simplicial cone generated by the pure diagrams of these elements, except \( f = (\ldots, r - 1, r + 1, \ldots). \)
Equations of exterior facets of type 3

Example

\[ D : \cdots < (-1, 0, 1, 3) < (-1, 0, 2, 3) < (-1, 1, 2, 3) < \cdots \]

Equation of hyperplane \( h_{D,f} \) is the following diagram rotated 90° degrees counterclockwise.

\[
\begin{array}{cccccccccccc}
6 & 5 & 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 \\
\cdots & 5 & 0 & -3 & -4 & -3 & 0 & 5 & 12 & 21 & 32 & \cdots \\
\cdots & -12 & -5 & 0 & 3 & 4 & 3 & 0 & -5 & -12 & -21 & \cdots \\
\cdots & 21 & 12 & 5 & 0 & -3 & -4 & -3 & 0 & 5 & 12 & \cdots \\
\end{array}
\]

The numbers in the first row are the values of \((d - 1)(d + 3)\).
When $h$ is the equation of an exterior facet of the fan $F$. Show that $h(\beta) \geq 0$ for any Betti diagram $\beta$ of a Cohen-Macaulay module.
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\[
h(\beta) \geq 0 \text{ for any Betti diagram } \beta \text{ of a Cohen-Macaulay module.}
\]

This makes us conclude that \( B \subseteq |F| \).
There exists a vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with Hilbert polynomial $\chi_\mathcal{E}(d) = (d - 1)(d + 3)$ whose cohomology table is:

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\cdots$</th>
<th>-6</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dim_k H^2_\mathcal{E}(d)$</td>
<td>$\cdots$</td>
<td>21</td>
<td>12</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\dim_k H^1_\mathcal{E}(d)$</td>
<td>$\cdots$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>$\cdots$</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>5</td>
<td>12</td>
<td>21</td>
</tr>
</tbody>
</table>
A vector bundle $\mathcal{E}$ on $\mathbb{P}^m$ is said to have supernaturl cohomology if there are integers $z_1 > z_2 > \cdots > z_m$ such that:

1. The Hilbert polynomial

   $$\chi_{\mathcal{E}}(d) = c(d - z_1) \cdots (d - z_m).$$

2. In each column of the cohomology table there is at most one nonzero value.

**Theorem**

*For each sequence $z_1 > \cdots > z_m$ such a bundle exists.*
Pairing between betti diagrams and cohomology tables

For a module $M$ let $\beta_{ij} = \beta_{ij}(M)$ be its Betti numbers.

For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{n-1}$ let $\gamma_{ij} = H^i(\mathbb{P}^{n-1}, \mathcal{F}(j))$ be its cohomology numbers.
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Let $e \in \mathbb{Z}$ and $0 \leq \tau \leq n - 1$, and define $\gamma_{\leq i, d}$ to be $\gamma_{0, d} - \gamma_{1, d} + \cdots + (-1)^i \gamma_{i, d}$. Define the pairing $\langle \beta, \gamma \rangle_{e, \tau}$ as the expression:
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\[
\sum_{i < \tau, d \in \mathbb{Z}} (-1)^i \beta_{i,d} \gamma_{\leq i,-d} + \sum_{d \leq e} (-1)^\tau \beta_{\tau,d} \gamma_{\leq \tau,-d} + \sum_{d > e} (-1)^\tau \beta_{\tau,d} \gamma_{\leq \tau-1,-d} + \sum_{d \leq e+1} (-1)^{\tau+1} \beta_{\tau+1,d} \gamma_{\leq \tau,-d} + \sum_{d > e+1} (-1)^{\tau+1} \beta_{\tau+1,d} \gamma_{\leq \tau-1,-d} + \sum_{i > \tau+1} (-1)^i \beta_{i,d} \gamma_{\leq i,-d}.
\]
Theorem (Eisenbud, Schreyer)

For any module $M$ and any coherent sheaf $\mathcal{F}$ the pairing:

$$\langle \beta(M), \gamma(\mathcal{F}) \rangle_{e, \tau} \geq 0.$$
Conclusion

As we saw earlier in the example:

**Theorem**

The degree sequence corresponding to an exterior facet of type 3:

\[ f = (f_0 < f_1 < \cdots < f_{\tau} < f_{\tau+1} < \cdots < f_n). \]

Let \( E \) be the vector bundle on \( \mathbb{P}^{n-1} \) with supernatural cohomology and root sequence \(-f_0 > -f_1 > \cdots > -f_{\tau-1} > -f_{\tau+2} > \cdots > -f_n.\)
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As we saw earlier in the example:

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*Then the hyperplane equation of the exterior facet obtained by omitting the pure diagram \( \pi(f) \) is:*

\[ h_{D,f}(\beta) = \langle \beta, \gamma(E) \rangle_{e,\tau} \]

where \( e = f_\tau. \)
Conclusion

We may then finally conclude:

**Corollary**

\[ h_{D,f}(\beta) \geq 0 \text{ for all Betti diagrams } \beta = \beta(M). \]

Hence the cone of Betti diagrams \( B \) is inside the geometric realization \( |F| \) of the fan \( F \).