The Existence of Relative Pure Injective Envelopes

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The Existence of Relative Pure Injective Envelopes

- $S$-pure Exact Sequences
- $S$-pure Homological Dimensions
- $S$-Pure Injective Envelopes
The notion of purity was introduced by Cohn (1959) for left $R$-modules and by Łoś (1957) for abelian groups. In 1967, Kiełpiński has introduced the notion of relative $\Gamma$-purity and proved that any $R$-module possesses a relative $\Gamma$-pure injective envelope. Two years later, Warfield has proved that any $R$-module admits a pure injective envelope. Also, he introduced the notion of $S$-purity for any class $S$ of $R$-modules.
Def. Let $S$ be a class of $R$-modules. An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of $R$-modules and $R$-homomorphisms is called $S$-pure exact if for all $U \in S$ the induced $R$-homomorphism $\text{Hom}_R(U, B) \to \text{Hom}_R(U, C)$ is surjective.
### Def.

Let $S$ be a class of $R$-modules. An exact sequence

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of $R$-modules and $R$-homomorphisms is called **$S$-pure exact** if for all $U \in S$ the induced $R$-homomorphism $\text{Hom}_R(U, B) \rightarrow \text{Hom}_R(U, C)$ is surjective.

In this situation, $f, g, f(A)$ and $C$ are called **$S$-pure monomorphism**, **$S$-pure epimorphism**, **$S$-pure submodule** of $B$, and **$S$-pure homomorphic image** of $B$; respectively.
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Def.

An $R$-module $P$ is called $S$-pure projective (resp. $S$-copure projective) if for any $S$-pure exact sequence (resp. $S$-copure exact sequence) $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$, the induced $R$-homomorphism $\text{Hom}_R(P, B) \to \text{Hom}_R(P, C)$ is surjective.
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An $R$-module $E$ is called $S$-pure injective (resp. $S$-copure injective) if for any $S$-pure exact sequence (resp. $S$-copure exact sequence) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced $R$-homomorphism $\text{Hom}_R(B, E) \to \text{Hom}_R(A, E)$ is surjective.
The existence of relative pure injective envelopes

\section*{S-pure Exact Sequences}

\textbf{Def.}

An $R$-module $P$ is called \textit{S-} pure projective (resp. \textit{S-} copure projective) if for any \textit{S-}pure exact sequence (resp. \textit{S-}copure exact sequence) $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$, the induced $R$-homomorphism $\text{Hom}_R(P, B) \rightarrow \text{Hom}_R(P, C)$ is surjective.

An $R$-module $E$ is called \textit{S-} pure injective (resp. \textit{S-} copure injective) if for any \textit{S-}pure exact sequence (resp. \textit{S-}copure exact sequence) $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$, the induced $R$-homomorphism $\text{Hom}_R(B, E) \rightarrow \text{Hom}_R(A, E)$ is surjective.

Also, an $R$-module $F$ is called \textit{S-} pure flat (resp. \textit{S-} copure flat) if for any \textit{S-}pure exact sequence (resp. \textit{S-}copure exact sequence) $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$, the induced $R$-homomorphism $F \otimes_R A \rightarrow F \otimes_R B$ is injective.
Def.

An $R$-module $M$ is called cyclically-presented if it is isomorphic to a module of the form $R^n/G$ for some $n \in \mathbb{N}$ and some cyclic submodule $G$ of $R^n$.

An $R$-module $M$ is called cyclic cyclically-presented if it is isomorphic to a module of the form $R/Rr$ for some $r \in R$.

If $S$ is the class of all finitely presented (resp. cyclic cyclically-presented) $R$-modules, then $S$-purity is called purity (resp. RD-purity).

If $S$ is the class of all cyclically-presented $R$-modules, then $S$-purity is called cyclically purity.
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S-pure Exact Sequences

**Def.**

Let $\mathcal{F}$ (resp. $\mathcal{G}$) be a class of $R$-modules and $M$ an $R$-module. An $R$-homomorphism $\phi : F \to M$ (resp. $\phi : M \to G$) where $F \in \mathcal{F}$ (resp. $G \in \mathcal{G}$) is called an $\mathcal{F}$-precover (resp. a $\mathcal{G}$-preenvelope) of $M$ if for any $F' \in \mathcal{F}$ (resp. $G' \in \mathcal{G}$), the induced $R$-homomorphism $\text{Hom}_R(F', F) \to \text{Hom}_R(F', M)$ (resp. $\text{Hom}_R(G, G') \to \text{Hom}_R(M, G')$) is surjective.
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If $\phi : F \to M$ (resp. $\phi : M \to G$) is an $\mathcal{F}$-precover (resp. a $\mathcal{G}$-preenvelope) of $M$ and any $R$-homomorphism $f : F \to F$ (resp. $f : G \to G$) such that $\phi f = \phi$ (resp. $f \phi = \phi$) is an automorphism, then $\phi$ is called an $\mathcal{F}$-cover (resp. a $\mathcal{G}$-envelope) of $M$. 
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The class $\mathcal{F}$ (resp. $\mathcal{G}$) is called (pre)covering (resp. (pre)enveloping) if every $R$-module admits an $\mathcal{F}$-(pre)cover (resp. a $\mathcal{G}$-(pre)envelope).
Def.

We call a class $\mathcal{S}$ of $R$-modules **set-presentable** if it has a subset $\mathcal{S}^*$, with the property that for any $U \in \mathcal{S}$ there is a $U^* \in \mathcal{S}^*$ with $U \cong U^*$.

Any class of **finitely presented** $R$-modules which is **closed under isomorphisms** is set-presentable.
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**Def.**

We call a class $S$ of $R$-modules set-presentable if it has a subset $S^*$, with the property that for any $U \in S$ there is a $U^* \in S^*$ with $U \cong U^*$. Any class of finitely presented $R$-modules which is closed under isomorphisms is set-presentable.

**Warfield (1969)**

Let $S$ be a set-presentable class of finitely presented $R$-modules containing $R$. Then every $R$-module possesses an $S$-pure projective precover.

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$S$-pure Exact Sequences

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**Lemma**

Let $S$ be a class of $R$-modules. An $R$-module $M$ is $S$-pure flat if and only if $M^+$ is $S$-pure injective.
In what follows we denote the Pontryagin duality functor \( \text{Hom}_\mathbb{Z}(-, \mathbb{Q}/\mathbb{Z}) \) by \((-)^+\).

**Lemma**

Let \( S \) be a class of \( R \)-modules. An \( R \)-module \( M \) is \( S \)-pure flat if and only if \( M^+ \) is \( S \)-pure injective.

**Cor.**

Let \( S \) be a class of \( R \)-modules. Then the class of \( S \)-pure flat \( R \)-modules is covering.
Def.

For any two natural integers \( n, k \) and any \( R \)-homomorphism \( \mu : R^k \to R^n \), let \( \mu^t : R^n \to R^k \) denote the \( R \)-homomorphism given by the transpose of the matrix corresponding to \( \mu \). Let \( U \) be a finitely presented \( R \)-module and \( R^k \xrightarrow{\mu} R^n \xrightarrow{\pi} U \to 0 \) a finitely presentation of \( U \). Then, the Auslander transpose of \( U \) is defined by \( \text{tr} (U) := \text{coker} \, \mu^t \). It is unique up to projective direct summands.
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In what follows, for a class \( S \) of finitely presented \( R \)-modules, we denote the class \( \{ \text{tr}(U) | U \in S \} \) by \( \text{tr}(S) \).
Let $S$ be a class of finitely presented $R$-modules. Obviously, if $R \in S$, then $R \in \text{tr}(S)$. If $S$ is set-presentable, then $\text{tr}(S)$ has a subclass $\tilde{S}$, which is a set and $\text{tr}(S)$-purity coincides with $\tilde{S}$-purity.
Let $S$ be a class of finitely presented $R$-modules. Obviously, if $R \in S$, then $R \in \text{tr}(S)$. If $S$ is set-presentable, then $\text{tr}(S)$ has a subclass $\tilde{S}$, which is a set and $\text{tr}(S)$-purity coincides with $\tilde{S}$-purity.

- If $S$ is the class of all cyclic free $R$-modules, then $S$-pure exact sequences are the usual exact sequences. So, $S$-pure projective, $S$-pure injective and $S$-pure flat $R$-modules are the usual projective, injective and flat $R$-modules; respectively. Moreover, $S = \text{tr}(S)$. 
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- If $S$ is the class of all finitely presented $R$-modules or the class of all cyclic cyclically-presented $R$-modules then $S = \text{tr}(S)$.
Let $\mathcal{S}$ be a class of finitely presented $R$-modules. Obviously, if $R \in \mathcal{S}$, then $R \in \text{tr}(\mathcal{S})$. If $\mathcal{S}$ is set-presentable, then $\text{tr}(\mathcal{S})$ has a subclass $\tilde{\mathcal{S}}$, which is a set and $\text{tr}(\mathcal{S})$-purity coincides with $\tilde{\mathcal{S}}$-purity.

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- If $\mathcal{S}$ is the class of all finitely presented $R$-modules or the class of all cyclic cyclically-presented $R$-modules then $\mathcal{S} = \text{tr}(\mathcal{S})$.
- If $\mathcal{S}$ is the class of all cyclically-presented $R$-modules, then $\tilde{\mathcal{S}} = \{R/I | I \text{ is a finitely generated ideal of } R\}$. 

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The Existence of Relative Pure Injective Envelopes

S-pure Exact Sequences

Let $S$ be a class of finitely presented $R$-modules. Obviously, if $R \in S$, then $R \in \text{tr}(S)$. If $S$ is set-presentable, then $\text{tr}(S)$ has a subclass $\tilde{S}$, which is a set and $\text{tr}(S)$-purity coincides with $\tilde{S}$-purity.

- If $S$ is the class of all cyclic free $R$-modules, then $S$-pure exact sequences are the usual exact sequences. So, $S$-pure projective, $S$-pure injective and $S$-pure flat $R$-modules are the usual projective, injective and flat $R$-modules; respectively. Moreover, $S = \text{tr}(S)$.
- If $S$ is the class of all finitely presented $R$-modules or the class of all cyclic cyclically-presented $R$-modules then $S = \text{tr}(S)$.
- If $S$ is the class of all cyclically-presented $R$-modules, then $\tilde{S} = \{R/I | I$ is a finitely generated ideal of $R\}$. 
Let $S$ be a class of finitely presented $R$-modules. Obviously, if $R \in S$, then $R \in \text{tr}(S)$. If $S$ is set-presentable, then $\text{tr}(S)$ has a subclass $\tilde{S}$, which is a set and $\text{tr}(S)$-purity coincides with $\tilde{S}$-purity.

- If $S$ is the class of all cyclic free $R$-modules, then $S$-pure exact sequences are the usual exact sequences. So, $S$-pure projective, $S$-pure injective and $S$-pure flat $R$-modules are the usual projective, injective and flat $R$-modules; respectively. Moreover, $S = \text{tr}(S)$.
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- If $S$ is the class of all cyclically-presented $R$-modules, then $\tilde{S} = \{ R/I | I \text{ is a finitely generated ideal of } R \}$.
Prop.

Let $S$ be a class of finitely presented $R$-modules and $E = 0 \rightarrow A \overset{i}{\rightarrow} B \overset{\psi}{\rightarrow} C \rightarrow 0$ an exact sequence of $R$-modules and $R$-homomorphisms. The following are equivalent:

1. $E$ is $S$-pure exact.
2. $\text{tr}(U) \otimes_R E$ is exact for all $U \in S$.
3. $\mu(A^k) = A^n \cap \mu(B^k)$ for all matrices $\mu \in \text{Hom}_R(R^k, R^n)$ with $\text{coker} \mu^t \in S$.
4. for any matrix $(r_{ij}) \in \text{Hom}_R(R^n, R^k)$ with $\text{coker} (r_{ij}) \in S$ and any $a_1, \ldots, a_n \in A$ if the linear equations $\sum_{i=1}^k r_{ij}x_i = a_j; 1 \leq j \leq n$ are soluble in $B$, then they are also soluble in $A$. 

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Lemma

Let $S$ be a class of $R$-modules and $X = \cdots \to X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \to \cdots$ an exact complex of $R$-modules. For each $i \in \mathbb{Z}$, set $X_i := 0 \to \text{Im} d_{i+1} \hookrightarrow X_i \to \text{Im} d_i \to 0$. Then

1. $\text{Hom}_R(U, X)$ is exact for all $U \in S$ if and only if $X_i$ is $S$-pure exact for all $i \in \mathbb{Z}$.

2. $\text{Hom}_R(X, V)$ is exact for all $V \in S$ if and only if $X_i$ is $S$-copure exact for all $i \in \mathbb{Z}$.
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**S-pure Exact Sequences**

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**Lemma**

Let $S$ be a class of $R$-modules and $\mathbf{X} = \cdots \rightarrow X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \rightarrow \cdots$ an exact complex of $R$-modules. For each $i \in \mathbb{Z}$, set $X_i := 0 \rightarrow \text{Im} \ d_{i+1} \hookrightarrow X_i \rightarrow \text{Im} \ d_i \rightarrow 0$. Then

1. $\text{Hom}_R(U, \mathbf{X})$ is exact for all $U \in S$ if and only if $X_i$ is $S$-pure exact for all $i \in \mathbb{Z}$.

2. $\text{Hom}_R(\mathbf{X}, V)$ is exact for all $V \in S$ if and only if $X_i$ is $S$-copure exact for all $i \in \mathbb{Z}$.

---

**Def.**

Let $S$ be a class of $R$-modules. An exact complex $\mathbf{X}$ of $R$-modules is said to be $S$-pure exact (resp. $S$-copure exact) if it satisfies the equivalent conditions of part (1) (resp. (2)) of the above Lemma.
Lemma

Let $S$ be a class of finitely presented $R$-modules and

$$X = \cdots \xrightarrow{d_{i+2}} X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \xrightarrow{d_{i-1}} \cdots$$

an exact complex of $R$-modules and $R$-homomorphisms. Then the following conditions are equivalent:

1. $X$ is $S$-pure exact.
2. $\text{Hom}_R(P, X)$ is exact for all $S$-pure projective $R$-modules $P$.
3. $\text{Hom}_R(X, E)$ is exact for all $S$-pure injective $R$-modules $E$.
4. $F \otimes_R X$ is exact for all $S$-pure flat $R$-modules $F$. 
Lemma

Let $S$ be a class of finitely presented $R$-modules and

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1. $X$ is $S$-pure exact.
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3. $\text{Hom}_R(X, E)$ is exact for all $S$-pure injective $R$-modules $E$.
4. $F \otimes_R X$ is exact for all $S$-pure flat $R$-modules $F$. 
Prop. Let $S$ be a set-presentable class of finitely presented $R$-modules containing $R$. Then, every $R$-module $M$ admits an $S$-pure injective preenvelope.
Prop.
Let $S$ be a set-presentable class of finitely presented $R$-modules containing $R$. Then, every $R$-module $M$ admits an $S$-pure injective preenvelope.

Cor.
Let $S$ be a set-presentable class of finitely presented $R$-modules containing $R$. Then, an $R$-module $E$ is $S$-pure injective if and only if it is isomorphic to a direct summand of a direct product of elements in $\text{tr}(S^*)^+ = \{ \text{tr}(U)^+ | U \in S^* \}$. 
Let $M$ be an $R$-module and $\mathcal{F}$ (resp. $\mathcal{G}$) a class of $R$-modules. A left $\mathcal{F}$ (resp. right $\mathcal{G}$)-resolution of $M$ is a Hom$_R(\mathcal{F}, -)$ (resp. Hom$_R(-, \mathcal{G})$) exact (not necessarily exact) complex

$$F_\bullet = \cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

(resp.

$$G_\bullet = 0 \rightarrow M \rightarrow G^0 \rightarrow \cdots \rightarrow G^n \rightarrow G^{n+1} \rightarrow \cdots$$

with $F_n \in \mathcal{F}$ (resp. $G^n \in \mathcal{G}$) for all $n \geq 0$. 

Def.

Let $M$ be an $R$-module and $\mathcal{F}$ (resp. $\mathcal{G}$) a class of $R$-modules. A left $\mathcal{F}$ (resp. right $\mathcal{G}$)-resolution of $M$ is a \(\text{Hom}_R(\mathcal{F}, -)\) (resp. $\text{Hom}_R(-, \mathcal{G})$) exact (not necessarily exact) complex

\[
\mathcal{F} \cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0
\]

(resp.

\[
\mathcal{G} \cdots \rightarrow 0 \rightarrow M \rightarrow G^0 \rightarrow \cdots \rightarrow G^n \rightarrow G^{n+1} \rightarrow \cdots
\]

with $F_n \in \mathcal{F}$ (resp. $G^n \in \mathcal{G}$) for all $n \geq 0$.

Let $\mathcal{F}$ be a precovering (resp. $\mathcal{G}$ be a preenveloping) class of $R$-modules. Then every $R$-module has a left $\mathcal{F}$ (resp. right $\mathcal{G}$)-resolution.
Def.

Let $\mathcal{F}$ and $\mathcal{G}$ be two classes of $R$-modules. The functor $\text{Hom}_R(−, \sim)$ is called right balanced by $\mathcal{F} \times \mathcal{G}$, if for each $R$-module $M$ there exists a $\text{Hom}_R(−, \mathcal{G})$ exact complex

$$F_\bullet = \cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

with $F_n \in \mathcal{F}$ for all $n \geq 0$ and a $\text{Hom}_R(\mathcal{F}, \sim)$ exact complex

$$G_\bullet = 0 \rightarrow M \rightarrow G^0 \rightarrow \cdots \rightarrow G^n \rightarrow G^{n+1} \rightarrow \cdots$$

with $G^n \in \mathcal{G}$ for all $n \geq 0$. 
Let $\mathcal{F}$ be a precovering class and $\mathcal{G}$ a preenveloping class of $R$-modules. Assume that

$$F_\bullet = \cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

is a left $\mathcal{F}$-resolution of an $R$-module $M$ and

$$G^\bullet = 0 \rightarrow N \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots \rightarrow G^n \rightarrow \cdots$$

is a right $\mathcal{G}$-resolution of an $R$ module $N$. Set:

$$F_\circ := \cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$$

and

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Let $\mathcal{F}$ be a precovering class and $\mathcal{G}$ a preenveloping class of $R$-modules. Assume that

$$F_\bullet = \cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

is a left $\mathcal{F}$-resolution of an $R$-module $M$ and

$$G^\bullet = 0 \to N \to G^0 \to G^1 \to \cdots \to G^n \to \cdots$$

is a right $\mathcal{G}$-resolution of an $R$ module $N$. Set:

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and

$$G^\circ := 0 \to G^0 \to G^1 \to \cdots \to G^n \to \cdots$$

Define $\text{Ext}^i_{\mathcal{F}}(M, N) := H^i(\text{Hom}_R(F_\circ, N))$ and $\text{Ext}^i_{\mathcal{G}}(M, N) := H^i(\text{Hom}_R(M, G^\circ))$ for all $i \geq 0$. 
The concept of pure homological dimensions was introduced in a special case by Griffith (1970), and in a general setting by Kiełpiński and Simson (1975).
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Let $S$ be a class of $R$-modules with the property that the class of $S$-pure projective $R$-modules is precovering and the class of $S$-pure injective $R$-modules is preenveloping. Denote the class of all $S$-pure projective (resp. $S$-pure injective) $R$-modules by $S\mathcal{P}$ (resp. $S\mathcal{I}$).
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For an $R$-module $M$, we define the $S$-pure projective dimension (resp. $S$-pure injective dimension) of $M$ as the infimum of the lengths of left $\mathcal{SP}$ (resp. right $\mathcal{SI}$)-resolutions of $M$. 
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Then, the global $S$-pure projective (resp. injective) dimension of $R$ is defined to be the supremum of the $S$-pure projective (resp. injective) dimensions of all $R$-modules.
Cor.

Let $S$ be a set-presentable class of finitely presented $R$-modules containing $R$. Denote the class of all $S$-pure projective (resp. $S$-pure injective) $R$-modules by $SP$ (resp. $SI$). Then the functor $\text{Hom}_R(-, \sim)$ is right balanced by $SP \times SI$. Consequently, $\text{Ext}^n_{SP}(M, N) \cong \text{Ext}^n_{SI}(M, N)$ for all $R$-modules $M$ and $N$ and all $n \geq 0$. Accordingly, the global $S$-pure projective dimension of $R$ is equal to its global $S$-pure injective dimension.
Thm.

Let $S$ be a set-presentable class of finitely presented $R$-modules containing $R$. Then every $R$-module $M$ possesses an $S$-pure injective envelope.
The Existence of Relative Pure Injective Envelopes

Thank You