On vertex decomposable simplicial complexes and their Alexander duals

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The 10th Seminar on Commutative Algebra and Related Topics
IPM, 18-19 December 2013
History

Pure k-decomposable simplicial complexes, first were defined by Provan and Billera for pure simplicial complexes, in connection with their study of diameter problems for pure complexes.

For 0-decomposable simplicial complexes (known as vertex decomposable), the definition was extended to non-pure complexes for by Björner and Wachs to obtain a new class of shellable simplicial complexes.
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For 0-decomposable simplicial complexes (known as vertex decomposable), the definition was extended to non-pure complexes for by Björner and Wachs to obtain a new class of shellable simplicial complexes.

Extension of k-decomposability to non-pure complexes was introduced by Woodroofe in order to study the independence complex of a chordal clutter.
History

▶ Pure k-decomposable simplicial complexes, first were defined by Provan and Billera for pure simplicial complexes, in connection with their study of diameter problems for pure complexes.

▶ For 0-decomposable simplicial complexes (known as vertex decomposable), the definition was extended to non-pure complexes for by Björner and Wachs to obtain a new class of shellable simplicial complexes.

▶ Extension of k-decomposability to non-pure complexes was introduced by Woodroofe in order to study the independence complex of a chordal clutter.
Defined in a recursive manner, vertex decomposable simplicial complexes form a well-behaved class of simplicial complexes. In many research papers vertex decomposability was used in an interesting way to study the algebraic properties of monomial ideals and nice results on edge ideals were obtained by combinatorial topological techniques.
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Minimal graded free resolution

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ with $\deg(x_i) = 1$

Any finitely generated graded $R$-module $M$ (such as a homogenous ideal $I$) has a minimal graded free resolution of length at most $n$, which can be presented as follows:

$$0 \to \bigoplus_{j \geq 0} R(-j)^{\beta_{p,j}} \to \cdots \to \bigoplus_{j \geq 0} R(-j)^{\beta_{1,j}} \to \bigoplus_{j \geq 0} R(-j)^{\beta_{0,j}} \to M \to 0$$
Minimal graded free resolution

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Basic facts

The numbers $\beta_{i,j}$ are uniquely determined by $M$:

$$\beta_{i,j}(M) = \dim_k \text{Tor}_i^R(M, k)_j$$

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A particularly simple sort of graded module is provided by the ideals generated by monomials, which are called monomial ideals. However, despite the misleading appearance of simplicity, it is still an open problem to describe explicitly the graded Betti numbers even in this case.

An explicit minimal resolution for a family of monomial ideals, which are called stable ideals, has been given by Eliahou and Kervaire.
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An explicit minimal resolution for a family of monomial ideals, which are called stable ideals, has been given by Eliahou and Kervaire.
Betti splitting

**Eliahou-Kervaire Splitting:** Let $I$, $J$ and $K$ be monomial ideals such that $G(I)$, the unique set of minimal generators of $I$, is the disjoint union of $G(J)$ and $G(K)$. Then $I = J + K$ is an **Eliahou-Kervaire splitting** if there exists a splitting function $G(J \cap K) \to G(J) \times G(K)$ sending $w \to (\varphi(w), \phi(w))$ such that

1. $w = \text{lcm}(\varphi(w), \phi(w))$ for all $w \in G(J \cap K)$, and

2. for every subset $S \subseteq G(J \cap K)$, $\text{lcm}(\varphi(S))$ and $\text{lcm}(\phi(S))$ strictly divide $\text{lcm}(S)$. 
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Betti splitting

[Fatabbi]. When $I = J + K$ is an Eliahou-Kervaire splitting,

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$

for all $i \in \mathbb{N}$ and (multi)degrees $j$. 
Betti splitting

There are other conditions on I, J and K, beyond the criterion of Eliahou and Kervaire, that imply that formula for Betti numbers holds. Consider the ideal

\[ I = (x_1x_2x_3, x_1x_3x_5, x_1x_4x_5, x_2x_3x_4, x_2x_4x_5). \]

There is no Eliahou and Kervaire splitting of I, but there are many ways to partition the minimal generators of I to form smaller ideals J and K so that the formula for Betti numbers still holds. Set

\[ I = (x_1x_2x_3, x_1x_3x_5, x_1x_4x_5) + (x_2x_3x_4, x_2x_4x_5). \]
**Betti splitting**

**Definition** *Francisco, Ha, Van Tuyl*. Let $I$, $J$ and $K$ be monomial ideals such that $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Then $I = J + K$ is a **Betti splitting** if

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\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)
$$

for all $i \in \mathbb{N}$ and (multi)degrees $j$. 
Betti splitting

[Francisco, Ha, Van Tuyl]. Let I, J and K be monomial ideals such that $I = J + K$ and $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Considering

$$0 \rightarrow J \cap K \rightarrow J \oplus K \rightarrow J + K = I \rightarrow 0$$

(1)

the following are equivalent:

1. $I = J + K$ is a Betti splitting.

2. the map $\text{Tor}_i(k, J \cap K)_j \rightarrow \text{Tor}_i(k, J)_j \oplus \text{Tor}_i(k, K)_j$ in the long exact sequence in Tor induced from (1) is the zero map.

3. applying the mapping cone construction to (1) gives a minimal free resolution of I.
Betti splitting

In general the mapping cone construction applied to (1) produces a free resolution of $I$ that is not necessarily minimal. In particular, the mapping cone construction implies that

$$\beta_{i,j}(I) \leq \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$
If $I = J + K$ is a Betti splitting, then

\[ \text{reg}(I) = \max \{ \text{reg}(J), \text{reg}(K), \text{reg}(J \cap K) - 1 \} \]
If $I = J + K$ is a Betti splitting, then

- $\text{reg}(I) = \max\{\text{reg}(J), \text{reg}(K), \text{reg}(J \cap K) - 1\}$
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Alexander dual of a simplicial complex and an ideal

For a simplicial complex $\Delta$ with the vertex set $X$, the Alexander dual simplicial complex $\Delta^\vee$ of $\Delta$ is defined as follows:

$$\Delta^\vee = \{ F \subseteq X; X \setminus F \notin \Delta \}$$

For a squarefree monomial ideal $I = (x_{1,1}x_{1,2} \cdots x_{1,k_1}, \ldots, x_{n,1}x_{n,2} \cdots x_{n,k_n})$, Alexander dual ideal of $I$ is defined as:

$$I^\vee = (x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}) \cap \cdots \cap (x_{n,1}, x_{n,2}, \ldots, x_{n,k_n})$$
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$$I_{\Delta^\vee} = (I_{\Delta})^\vee = (x^F_c : F \text{ is a facet of } \Delta)$$
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Alexander dual ideal

\[ \Delta = \langle \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{1, 5\}, \{2, 5\} \rangle \]

\[ N(\Delta) = \{\{1, 2, 5\}, \{1, 3, 5\}, \{2, 4, 5\}, \{2, 3, 5\}, \{1, 4\}\} \]

\[ I_\Delta = (x_1 x_2 x_5, x_1 x_3 x_5, x_2 x_4 x_5, x_2 x_3 x_5, x_1 x_4) \]

\[ I_{\Delta^\vee} = (x_4 x_5, x_1 x_5, x_1 x_2, x_2 x_3 x_4, x_1 x_3 x_4) = \]

\[ (x_1, x_2, x_5) \cap (x_1, x_3, x_5) \cap (x_2, x_4, x_5) \cap (x_2, x_3, x_5) \cap (x_1, x_4) \]
Ideals with linear quotients were defined by Herzog and Takayama in connection to their work on minimal free resolution of monomial ideals. A monomial ideal \( I = (f_1, \ldots, f_m) \) has linear quotients, if there exists an order \( f_1 < \cdots < f_m \) on the minimal generators of \( I \) such that the colon ideal \((f_1, \ldots, f_{i-1}) : f_i\) is generated by a subset of variables for all \( 2 \leq i \leq m \).
Alexander dual concepts

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[Herzog-Hibi-Zheng]. Let $\Delta$ be a simplicial complex and $I = I_\Delta$. Then

$\Delta$ is shellable $\iff I^\vee$ has linear quotients
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[Herzog-Hibi-Zheng]. Let \( \Delta \) be a simplicial complex and \( I = I_\Delta \). Then

\[ \Delta \text{ is shellable} \iff I^\vee \text{ has linear quotients} \]
Alexander dual concepts

[Eagon-Reiner]. Let $\Delta$ be a simplicial complex. Then

$$k[\Delta] \text{ is Cohen–Macaulay} \iff I_{\Delta^\vee} \text{ has linear resolution}$$

[Herzog-Hibi]. $k[\Delta]$ is sequentially Cohen-Macaulay if and only if $I_{\Delta^\vee}$ is componentwise linear.
Alexander dual concepts

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[Terai]. For a simplicial complex $\Delta$, $\text{pd}(I_{\Delta}) = \text{reg}(R/I_{\Delta^\vee})$. 
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[Terai]. For a simplicial complex $\Delta$, $pd(I_{\Delta}) = reg(R/I_{\Delta^\vee})$. 
**Vertex decomposable simplicial complex**

Let $\Delta$ be a simplicial complex and $F \in \Delta$. The deletion of $F$ is defined as:

$$\text{del}_\Delta(F) = \{ G \in \Delta : G \cap F = \emptyset \}$$

and the link of $F$

$$\text{lk}(F) = \{ G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta \}$$

Let $\Delta$ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$. Then $\Delta$ is vertex decomposable if either:

1. The only facet of $\Delta$ is $\{x_1, \ldots, x_n\}$, or $\Delta = \emptyset$.

2. There exists a vertex $x \in V$ such that $\text{del}_\Delta(x)$ and $\text{lk}_\Delta(x)$ are vertex decomposable, and such that every facet of $\text{del}_\Delta(x)$ is a facet of $\Delta$. 
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The vertex $x$ is called a shedding vertex for $\Delta$. 
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\[ \Delta \quad : \quad \text{del}_\Delta(4) \quad : \quad \text{lk}_\Delta(4) \]

\[ \Delta' = \text{lk}_\Delta(4) \quad : \quad \text{del}_{\Delta'}(1) \quad : \quad \text{lk}_{\Delta'}(1) \]
If $\Delta$ is vertex decomposable with shedding vertex $x$, then

$$\Delta = \text{del}_\Delta(x) \cup (\text{lk}_\Delta(x) \ast \{x\}).$$

What is the properties of Alexander dual ideal of a vertex decomposable simplicial complex?
If $\Delta$ is vertex decomposable with shedding vertex $x$, then

$$\Delta = \text{del}_\Delta(x) \cup (\text{lk}_\Delta(x) \ast \{x\}).$$

What is the properties of Alexander dual ideal of a vertex decomposable simplicial complex?
A monomial ideal $I$ of $R$ is called **vertex splittable** if it can be obtained by the following recursive procedure:

(i) If $u$ is a monomial and $I = (u)$, $I = (0)$ or $I = R$, then $I$ is a vertex splittable ideal.

(ii) If there is a variable $x \in X$ and vertex splittable ideals $I_1$ and $I_2 \subseteq k[X \setminus \{x\}]$ so that $I = xl_1 + l_2$, $l_2 \subseteq l_1$ and $G(I)$ is the disjoint union of $G(xl_1)$ and $G(l_2)$, then $I$ is a vertex splittable ideal.

With the above notations if $I = xl_1 + l_2$ is a vertex splittable ideal, then $xl_1 + l_2$ is called a **vertex splitting** for $I$. 
A monomial ideal \( I \) of \( R \) is called vertex splittable if it can be obtained by the following recursive procedure:

(i) If \( u \) is a monomial and \( I = (u) \), \( I = (0) \) or \( I = R \), then \( I \) is a vertex splittable ideal.

(ii) If there is a variable \( x \in X \) and vertex splittable ideals \( I_1 \) and \( I_2 \subseteq k[X \setminus \{x\}] \) so that \( I = xI_1 + I_2 \), \( I_2 \subseteq I_1 \) and \( G(I) \) is the disjoint union of \( G(xI_1) \) and \( G(I_2) \), then \( I \) is a vertex splittable ideal.

With the above notations if \( I = xI_1 + I_2 \) is a vertex splittable ideal, then \( xI_1 + I_2 \) is called a vertex splitting for \( I \).
Vertex splittable ideal

Example.

Let $I = (x_1^2x_2^3, x_3x_5^2, x_1x_2x_5, x_1x_3x_5, x_1x_4x_5, x_2x_5^3x_6)$. Then

$$I = x_1(x_1x_2^3, x_2x_5, x_3x_5, x_4x_5) + (x_3x_5^2, x_2x_5^3x_6)$$

and

$$(x_3x_5^2, x_2x_5^3x_6) \subseteq (x_1x_2^3, x_2x_5, x_3x_5, x_4x_5)$$

$$I_1 = (x_1x_2^3, x_2x_5, x_3x_5, x_4x_5) = x_5(x_2, x_3, x_4) + (x_1x_2^3)$$

$$x_2, x_3, x_4 = x_2(1) + (x_3, x_4)$$

$$I_2 = (x_3x_5^2, x_2x_5^3x_6) = x_3(x_5^2) + (x_2x_5^3x_6)$$

$I = (x_1x_3, x_2x_4) \text{ is not vertex splittable.}$
Vertex splittable ideal

Example.

Let \( I = (x_1^2 x_2^3, x_3 x_5^2, x_1 x_2 x_5, x_1 x_3 x_5, x_1 x_4 x_5, x_2 x_3^3 x_6) \). Then

\[
I = x_1 (x_1 x_2^3, x_2 x_5, x_3 x_5, x_4 x_5) + (x_3 x_5^2, x_2 x_3^3 x_6)
\]

and

\[
(x_3 x_5^2, x_2 x_3^3 x_6) \subseteq (x_1 x_2^3, x_2 x_5, x_3 x_5, x_4 x_5)
\]

\[
l_1 = (x_1 x_2^3, x_2 x_5, x_3 x_5, x_4 x_5) = x_5 (x_2, x_3, x_4) + (x_1 x_2^3)
\]

\[
(x_2, x_3, x_4) = x_2 (1) + (x_3, x_4)
\]

\[
l_2 = (x_3 x_5^2, x_2 x_3^3 x_6) = x_3 (x_5^2) + (x_2 x_3^3 x_6)
\]

\[
I = (x_1 x_3, x_2 x_4) \text{ is not vertex splittable.}
\]
Properties of vertex splittable ideals

**Theorem.** A simplicial complex $\Delta$ is vertex decomposable if and only if $I_{\Delta^\vee}$ is a vertex splittable ideal.

**Theorem.** Any vertex splittable ideal has linear quotients.
Properties of vertex splittable ideals

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**Theorem.** Any vertex splittable ideal has linear quotients.

**Outline of proof:** Let $I = xI_1 + I_2$, where $I_1$ and $I_2$ are vertex splittable. By induction, let $f_1 < \cdots < f_r$ and $g_1 < \cdots < g_s$ be the order of linear quotients on the minimal generators of $I_1$ and $I_2$, respectively. The ordering

$$xf_1 < \cdots < xf_r < g_1 < \cdots < g_s$$

is an order of linear quotients on the minimal generators of $I$. 
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Example. \( I = x_1(x_1x_2^3, x_2x_5, x_3x_5, x_4x_5) + (x_3x_5^2, x_2x_5^3x_6) \)

\( x_2x_5 < x_3x_5 < x_4x_5 < x_1x_2^3 \) and \( x_3x_5^2 < x_2x_5^3x_6 \) are order of linear quotients.

\[
\downarrow
\]

\[
x_1x_2x_5 < x_1x_3x_5 < x_1x_4x_5 < x_1^2x_2^3 < x_3x_5^2 < x_2x_5^3x_6
\]

is an order of linear quotients for \( I \).

Corollary. Let \( I \) be a vertex splittable ideal generated by monomials in the same degrees. Then \( I \) has a linear resolution. \( \Delta \) pure + vertex decomposable \( \Rightarrow \Delta \) is Cohen-Macaulay
Properties of vertex splittable ideals

Example. \( I = x_1(x_1x_2^3, x_2x_5, x_3x_5, x_4x_5) + (x_3x_5^2, x_2x_5^3x_6) \)
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(\( \Delta \) pure + vertex decomposable \( \Rightarrow \) \( \Delta \) is Cohen-Macaulay)
Properties of vertex splittable ideals

Example. \( I = x_1(x_1x_2^3, x_2x_5, x_3x_5, x_4x_5) + (x_3x_5^2, x_2x_5^3x_6) \)
\( x_2x_5 < x_3x_5 < x_4x_5 < x_1x_2^3 \) and \( x_3x_5^2 < x_2x_5^3x_6 \) are order of linear quotients.

\[ \downarrow \]

\( x_1x_2x_5 < x_1x_3x_5 < x_1x_4x_5 < x_1^2x_2^3 < x_3x_5^2 < x_2x_5^3x_6 \)

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Corollary. Let \( I \) be a vertex splittable ideal generated by monomials in the same degrees. Then \( I \) has a linear resolution. (\( \Delta \) pure + vertex decomposable \( \Rightarrow \Delta \) is Cohen-Macaulay)
Resolution by mapping cone for ideals with linear quotients: [Herzog, Takayama].

$I$ : a monomial ideal with linear quotients with the ordering $f_1 < \cdots < f_m$ on its minimal generators

$l_j = (f_1, \ldots, f_j)$

$L_j = (u_1, \ldots, u_j) : u_{j+1}$

$l_{j+1}/l_j \cong R/L_j \Rightarrow$ we get the exact sequences

$$0 \to R/L_j \xrightarrow{u_{j+1}} R/l_j \to R/l_{j+1} \to 0$$
Betti splitting

Resolution by mapping cone for ideals with linear quotients:
[Herzog, Takayama].

$I$ : a monomial ideal with linear quotients with the ordering
$f_1 < \cdots < f_m$ on its minimal generators

$l_j = (f_1, \ldots, f_j)$
$L_j = (u_1, \ldots, u_j) : u_{j+1}$

$l_{j+1}/l_j \cong R/L_j \Rightarrow$ we get the exact sequences

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$F(j) :$ minimal graded free resolution of $R/I_j$

$K(j) :$ the Koszul complex for the regular sequence $x_{k_1}, \ldots, x_{k_l}$,
where set$(u_{j+1}) = \{x_{k_1}, \ldots, x_{k_l}\}$. 

Betti splitting

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Betti splitting

\[ \psi(j) : K(j) \to F(j) \text{ a graded complex homomorphism lifting} \]
\[ R/L_j \to R/I_j \]

The mapping cone \( C(\psi(j)) \) of \( \psi(j) \) yields a minimal graded free resolution of \( R/I_{j+1} \).

By iterated mapping cones one obtains step by step a minimal graded free resolution of \( R/I \).

\[ \beta_{i,j}(I) = \sum_{\deg(f_t) = j - i} \binom{|\text{set}_I(f_t)|}{i}. \]
Betti splitting

$I$ : a vertex splittable ideal with vertex splitting $I = xI_1 + I_2$

$$\beta_{i,j}(I) = \sum_{\text{deg}(f_t) = j-i} \binom{|\text{set}_I(f_t)|}{i}.$$ 

$f_1 < \cdots < f_r$ : the order of linear quotients on the minimal generators of $I_1$

$g_1 < \cdots < g_s$ : the order of linear quotients on the minimal generators of $I_2$

$xf_1 < \cdots < xf_r < g_1 < \cdots < g_s$

is an order of linear quotients for $I$, 

$\set(I(f_t)) = \set(I_1(f_t))$ (1 \leq t \leq r)

and

$\set(I(g_k)) = \{x\} \cup \set(I_2(g_k))$ (1 \leq k \leq s)
Betti splitting

$I$: a vertex splittable ideal with vertex splitting $I = xl_1 + l_2$

$$\beta_{i,j}(I) = \sum_{\deg(f_t) = j - i} \binom{|\text{set}_I(f_t)|}{i}.$$ 

$f_1 < \cdots < f_r$: the order of linear quotients on the minimal generators of $l_1$

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$xf_1 < \cdots < xf_r < g_1 < \cdots < g_s$

is an order of linear quotients for $I$,

$$\text{set}_I(xf_t) = \text{set}_{l_1}(f_t) \quad (1 \leq t \leq r)$$

and

$$\text{set}_I(g_k) = \{x\} \cup \text{set}_{l_2}(g_k) \quad (1 \leq k \leq s)$$
Betti splitting

$I$: a vertex splittable ideal with vertex splitting $I = xI_1 + I_2$

$$\beta_{i,j}(I) = \sum_{\deg(f_t) = j - i} \binom{|\text{set}_{I}(f_t)|}{i}. $$

$f_1 < \cdots < f_r$: the order of linear quotients on the minimal generators of $I_1$

$g_1 < \cdots < g_s$: the order of linear quotients on the minimal generators of $I_2$

$xf_1 < \cdots < xf_r < g_1 < \cdots < g_s$

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and

$$\text{set}_I(g_k) = \{x\} \cup \text{set}_{I_2}(g_k) \quad (1 \leq k \leq s)$$
Betti splitting

\[ \beta_{i,j}(I) = \sum_{\deg(f_t)=j-i-1} (|\text{set}_I(xf_t)|_i) + \sum_{\deg(g_k)=j-i} (|\text{set}_I(g_k)|_i) \]

\[ \Downarrow \]

\[ \beta_{i,j}(I) = \sum_{\deg(f_t)=j-i-1} (|\text{set}_{I_1}(f_t)|_i) + \sum_{\deg(g_k)=j-i} (|\text{set}_{I_2}(g_k)|_i + 1) \]

Applying the equality

\[ (|\text{set}_{I_2}(g_k)|_i + 1) = (|\text{set}_{I_2}(g_k)|_i) + (|\text{set}_{I_2}(g_k)|_{i-1}) \]
Betti splitting

\[ \beta_{i,j}(I) = \sum_{\deg(f_t)=j-i-1} \binom{|\text{set}_I(xf_t)|}{i} + \sum_{\deg(g_k)=j-i} \binom{|\text{set}_I(g_k)|}{i} \]

\[ \Downarrow \]

\[ \beta_{i,j}(I) = \sum_{\deg(f_t)=j-i-1} \binom{|\text{set}_{I_1}(f_t)|}{i} + \sum_{\deg(g_k)=j-i} \binom{|\text{set}_{I_2}(g_k)| + 1}{i} \]

Applying the equality

\[ \binom{|\text{set}_{I_2}(g_k)| + 1}{i} = \binom{|\text{set}_{I_2}(g_k)|}{i} + \binom{|\text{set}_{I_2}(g_k)|}{i-1} \]
Betti splitting

\[\sum_{\deg(g_k) = j-i} \binom{|\text{set}_{l_2}(g_k)| + 1}{i} = \sum_{\deg(g_k) = j-i} \binom{|\text{set}_{l_2}(g_k)|}{i} + \sum_{\deg(g_k) = j-i} \binom{|\text{set}_{l_2}(g_k)|}{i-1}\]

\[= \beta_{i,j}(l_2) + \beta_{i-1,j-1}(l_2)\]

Also

\[\sum_{\deg(f_t) = j-i-1} \binom{|\text{set}_{l_1}(f_t)|}{i} = \beta_{i,j-1}(l_1)\]
Betti splitting

$$\beta_{i,j}(l) = \beta_{i,j-1}(l_1) + \beta_{i,j}(l_2) + \beta_{i-1,j-1}(l_2).$$

$l_2 \subseteq l_1$ implies that $xl_1 \cap l_2 = xl_2$. Also $\beta_{i,j-1}(l_1) = \beta_{i,j}(xl_1)$ and $\beta_{i-1,j-1}(l_2) = \beta_{i-1,j}(xl_2)$.

$$\Downarrow$$

$$\beta_{i,j}(l) = \beta_{i,j}(xl_1) + \beta_{i,j}(l_2) + \beta_{i-1,j}(xl_1 \cap l_2).$$
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\[ \beta_{i,j}(l) = \beta_{i,j-1}(l_1) + \beta_{i,j}(l_2) + \beta_{i-1,j-1}(l_2). \]

\( l_2 \subseteq l_1 \) implies that \( xl_1 \cap l_2 = xl_2 \). Also \( \beta_{i,j-1}(l_1) = \beta_{i,j}(xl_1) \) and \( \beta_{i-1,j-1}(l_2) = \beta_{i-1,j}(xl_2) \).

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\[ \beta_{i,j}(l) = \beta_{i,j}(xl_1) + \beta_{i,j}(l_2) + \beta_{i-1,j}(xl_1 \cap l_2). \]
**Theorem.** Let $I = xI_1 + I_2$ be a vertex splitting for the monomial ideal $I$. Then $I = xI_1 + I_2$ is a **Betti splitting**.

\[ 0 \rightarrow xI_2 \rightarrow xI_1 \oplus I_2 \rightarrow I \rightarrow 0 \]

gives a minimal free resolution of $I$.
Betti splitting

**Theorem.** Let \( I = xI_1 + I_2 \) be a vertex splitting for the monomial ideal \( I \). Then \( I = xI_1 + I_2 \) is a Betti splitting.

\[\downarrow\]

Applying the mapping cone to

\[ 0 \to xI_2 \to xI_1 \oplus I_2 \to I \to 0 \]

gives a minimal free resolution of \( I \).
Corollary. For a vertex splittable ideal $I$ with vertex splitting $I = xI_1 + I_2$, the graded Betti numbers of $I$ can be computed by the following recursive formula

$$\beta_{i,j}(I) = \beta_{i,j-1}(I_1) + \beta_{i,j}(I_2) + \beta_{i-1,j-1}(I_2).$$

Corollary. Let $\Delta$ be a vertex decomposable simplicial complex, $x$ a shedding vertex of $\Delta$, $\Delta_1 = \text{del}_\Delta(x)$ and $\Delta_2 = \text{lk}_\Delta(x)$. Then

$$\beta_{i,j}(I_{\Delta^\vee}) = \beta_{i,j-1}(I_{\Delta_1^\vee}) + \beta_{i,j}(I_{\Delta_2^\vee}) + \beta_{i-1,j-1}(I_{\Delta_2^\vee}).$$
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$I_\Delta^\vee = xI_{\Delta_1}^\vee + I_{\Delta_2}^\vee$ is a vertex splitting
Corollary. For a vertex splittable ideal \( I \) with vertex splitting \( I = xl_1 + l_2 \), the graded Betti numbers of \( I \) can be computed by the following recursive formula

\[
\beta_{i,j}(I) = \beta_{i,j-1}(l_1) + \beta_{i,j}(l_2) + \beta_{i-1,j-1}(l_2).
\]

Corollary. Let \( \Delta \) be a vertex decomposable simplicial complex, \( x \) a shedding vertex of \( \Delta \), \( \Delta_1 = \text{del}_\Delta(x) \) and \( \Delta_2 = \text{lk}_\Delta(x) \). Then

\[
\beta_{i,j}(l_\Delta \vee) = \beta_{i,j-1}(l_\Delta \vee_1) + \beta_{i,j}(l_\Delta \vee_2) + \beta_{i-1,j-1}(l_\Delta \vee_2).
\]

\( l_\Delta \vee = xl_\Delta \vee_1 + l_\Delta \vee_2 \) is a vertex splitting
**Corollary.** Let $\Delta$ be a vertex decomposable simplicial complex, $x$ a shedding vertex of $\Delta$ and $\Delta_1 = \text{del}_\Delta(x)$ and $\Delta_2 = \text{lk}_\Delta(x)$. Then

$$\text{pd}(R/I_\Delta) = \max\{\text{pd}(R/I_{\Delta_1}) + 1, \text{pd}(R/I_{\Delta_2})\},$$

$$\text{reg}(R/I_\Delta) = \max\{\text{reg}(R/I_{\Delta_1}), \text{reg}(R/I_{\Delta_2}) + 1\}.$$
Vertex cover ideal of a graph

Let $G = (V(G), E(G))$ be a graph. A subset $C \subseteq V(G)$ is called a **vertex cover** of $G$ if it intersects all the edges of $G$.

A vertex cover $C$ is a **minimal vertex cover** if no proper subset of $C$ is a vertex cover.
Vertex cover ideal of a graph

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Let $C_1, \ldots, C_k$ be the minimal vertex covers of $G$. Then

$$J(G) = (x^{C_i} : 1 \leq i \leq k)$$

is called the vertex cover ideal of $G$. 
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One has

$$J(G) = \text{I}(G)^\vee = \bigcap_{\{x_i, x_j\} \in E(G)} (x_i, x_j)$$
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$$J(G) = I(G)^\vee = \bigcap_{\{x_i, x_j\} \in E(G)} (x_i, x_j)$$
Vertex cover ideal of a vertex decomposable graph

A graph $G$ is called vertex decomposable if $\Delta_G$ is vertex decomposable.

If $\Delta = \Delta_G$ and $v \in V(G)$, then $\text{del}_\Delta(v) = \Delta_G \setminus \{v\}$ and $\text{lk}_\Delta(v) = \Delta_G \setminus N_G[v]$. 
A graph $G$ is called vertex decomposable if $\Delta_G$ is vertex decomposable.

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**Theorem.** Let $G$ be a vertex decomposable graph, $v \in V(G)$ be a shedding vertex of $G$, $G' = G \setminus \{v\}$, $G'' = G \setminus N_G[v]$ and $\deg_G(v) = t$. Then

$$\beta_{i,j}(J(G)) = \beta_{i,j-1}(J(G')) + \beta_{i,j-t}(J(G'')) + \beta_{i-1,j-t-1}(J(G'')).$$
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Let $G$ be a Cohen-Macaulay bipartite graph, $x, y \in V(G)$ be adjacent vertices with $\deg_G(x) = 1$ such that $G' = G \setminus N_G[x]$ and $G'' = G \setminus N_G[y]$ are Cohen-Macaulay and $\deg_G(y) = t$. Then

$$\beta_i(J(G)) = \beta_i(J(G')) + \beta_i(J(G'')) + \beta_{i-1}(J(G''))$$
Vertex cover ideal of a vertex decomposable graph

**Theorem.** Let $G$ be a *sequentially Cohen-Macaulay bipartite* graph, $x, y \in V(G)$ be adjacent vertices with $\text{deg}_G(x) = 1$ such that $G' = G \setminus N_G[x]$ and $G'' = G \setminus N_G[y]$ are sequentially Cohen-Macaulay and $\text{deg}_G(y) = t$. Then

$$\beta_{i,j}(J(G)) = \beta_{i,j-1}(J(G')) + \beta_{i,j-t}(J(G'')) + \beta_{i-1,j-t-1}(J(G''))$$

$$\{\text{sequentially Cohen-Macaulay bipartite graph}\} \subseteq \{\text{vertex decomposable graphs}\}$$

Also $y$ is a *shedding vertex* of $G$. 
Vertex cover ideal of a vertex decomposable graph

**Theorem.** Let $G$ be a sequentially Cohen-Macaulay bipartite graph, $x, y \in V(G)$ be adjacent vertices with $\deg_G(x) = 1$ such that $G' = G \setminus N_G[x]$ and $G'' = G \setminus N_G[y]$ are sequentially Cohen-Macaulay and $\deg_G(y) = t$. Then

$$\beta_{i,j}(J(G)) = \beta_{i,j-1}(J(G')) + \beta_{i,j-t}(J(G'')) + \beta_{i-1,j-t-1}(J(G''))$$

$$\{\text{sequentially Cohen-Macaulay bipartite graph}\} \subseteq \{\text{vertex decomposable graphs}\}$$

Also $y$ is a shedding vertex of $G$. 
Vertex cover ideal of a vertex decomposable graph

A graph $G$ is called chordal, if it contains no induced cycle of length 4 or greater.

In a graph $G$, a vertex $x$ is called a simplicial vertex if the induced subgraph on the set $N_G[x]$ is a complete graph.
A graph \( G \) is called **chordal**, if it contains no induced cycle of length 4 or greater.

In a graph \( G \), a vertex \( x \) is called a **simplicial vertex** if the induced subgraph on the set \( N_G[x] \) is a complete graph.
Vertex cover ideal of a vertex decomposable graph

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Vertex cover ideal of a vertex decomposable graph

[Dirac]. Any chordal graph has a simplicial vertex.

**Theorem.** Let $G$ be a chordal graph with simplicial vertex $x$ and $y \in N_G(x)$ with $\deg_G(y) = t$. Let $G' = G \setminus \{y\}$ and $G'' = G \setminus N_G[y]$. Then

$$\beta_{i,j}(J(G)) = \beta_{i,j-1}(J(G')) + \beta_{i,j-t}(J(G'')) + \beta_{i-1,j-t-1}(J(G''))$$
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Chordal $\Rightarrow$ vertex decomposable

A neighbour of a simplicial vertex is a shedding vertex.
Vertex cover ideal of a vertex decomposable graph

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**Chordal $\Rightarrow$ vertex decomposable**

A neighbour of a simplicial vertex is a shedding vertex.
Vertex splittable edge ideals

**Theorem.** Let $G$ be a chordal graph. Then $I(G^c)$ is a **vertex splittable ideal**.

$x \in V(G): a simplicial vertex
N_G(x) = \{x_1, \ldots, x_k\}$

$$I(G^c) = x(x_{k+1}, \ldots, x_n) + I(G_0^c)$$

, where $G_0 = G \setminus \{x\}$

$$I(G_0^c) \subseteq (x_{k+1}, \ldots, x_n)$$

$G_0$ is chordal $\Rightarrow$ $I(G_0^c)$ is vertex splittable

$(x_{k+1}, \ldots, x_n)$ is vertex splittable
Vertex splittable edge ideals

**Theorem.** Let $G$ be a chordal graph. Then $I(G^c)$ is a vertex splittable ideal.

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, where $G_0 = G \setminus \{x\}$

\[ I(G_0^c) \subseteq (x_{k+1}, \ldots, x_n) \]

$G_0$ is chordal $\Rightarrow$ $I(G_0^c)$ is vertex splittable

$(x_{k+1}, \ldots, x_n)$ is vertex splittable
Edge ideals with linear resolution

[Fröberg]. For a graph $G$, the edge ideal $I(G)$ has linear resolution if and only if $G^c$ is a chordal graph.

Corollary. For a graph $G$, the edge ideal $I(G)$ is vertex splittable if and only if $I(G)$ has linear resolution.
[Fröberg]. For a graph $G$, the edge ideal $I(G)$ has linear resolution if and only if $G^c$ is a chordal graph.

\[\Downarrow\]

Corollary. For a graph $G$, the edge ideal $I(G)$ is vertex splittable if and only if $I(G)$ has linear resolution.
REFERENCES

Thanks!