Two results on the regularity of monomial ideals

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Outline

1. Preliminaries
   - Castelnuovo-Mumford Regularity
   - Simplicial Complexes
2. On the growth of the degree of syzygies
3. Regularity via square-free components

$S = K[x_1, \ldots, x_n]$ polynomial ring over K.
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For every $i \in \mathbb{N} \cup \{0\}$, one defines:

$$t^S_i(M) = \max\{j : \quad \beta^K_{i,j}(M) \neq 0\},$$

and $t^S_i(M) = -\infty$, if it happens that $\operatorname{Tor}^S_i(K, M) = 0$.

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The *Castelnuovo-Mumford regularity* of M, $\operatorname{reg}(M)$, is given by:

$$\operatorname{reg}(M) = \sup\{t^S_i(M) - i : \quad i \in \mathbb{Z}\}.$$

initial degree of $0 \neq M = \oplus_{i \in \mathbb{Z}} M_i$:

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(ii) *$\beta^K_{i,i+j}(M) = 0$, for all $j \neq d$.*

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(iii) *The graded minimal free resolution of M is of the form,*

$$0 \to S^{\beta^K_\rho}(-d-\rho) \to \cdots \to S^{\beta^K_1}(-d-1) \to S^{\beta^K_0}(-d) \to M \to 0.$$

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(iv) *$\mathrm{reg}\,(M) = d$ and M has a minimal homogeneous generator all of degree d.*

Definition (Simplicial complex)

A *simplicial complex* Δ over a set of vertices $V = \{v_1, \ldots, v_n\}$, is a collection of subsets of V, with the property that:

(a) $\{v_i\} \in \Delta$, for all i;

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For $F \subset \{v_1, \ldots, v_n\}$, we set $\mathbf{x}_F = \prod_{v_i \in F} x_i$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ is defined as follows:

$$I_\Delta = \langle \mathbf{x}_F : \quad F \notin \Delta \rangle \subset K[x_1, \ldots, x_n]$$

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Theorem (Hochster formula)

Let Δ be a simplicial complex over $n = \{1, \ldots, n\}$, and K be a field. Then,

$$\beta^K_{i,j}(I_\Delta) = \sum_{\substack{W \subset [n] \\ |W| = j}} \dim_K \tilde{H}_{j-i-2}(\Delta_W; K).$$

Outline

Proposition (Y-Z (2013))

Let Δ *be a simplicial complex on vertex set* $[n]$ *and* $A = \{a_1, \dots, a_d\}$ *be a* d*-subset of* $[n]$ *such that* $A \notin \Delta$. *Let* $I := I_\Delta \subset K[x_1, \dots, x_n]$ *be the Stanley-Reisener ideal of* Δ *and* $i \geq 0, j \geq i + d$ *be non-negative integers with,*

$$\beta^K_{i,j}(I) = \beta^K_{i,j+1}(I) = \cdots = \beta^K_{i,j+(d-1)}(I) = 0.$$

If W *is a subset of* $[n]$ *with* $|W| = j + d$ *and* $A \subset W$, *then:*

$$\tilde{H}_{j-i+(d-t-3)}\left(\bigcup_{i=j_0}^{d} \Delta_{W \setminus \{a_1,\dots,a_t,a_i\}}; K\right) = 0$$

for all t, j_0, *with* $0 \leq t \leq d - 1$ *and* $t < j_0 \leq d$.

Theorem (Y-Z (2013))

Let Δ *be a simplicial complex on vertex set* $[n]$ *and* $I = I_\Delta \subset K[x_1, \ldots, x_n]$ *be the Stanley-Reisener ideal of* Δ. *Let* $d = t_0(I)$. *If* $i \geq 0$ *and* $j \geq i + d$ *be non-negative integers such that,*

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then, $\beta^K_{i+1,j+d}(I) = 0$.

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Corollary (Y-Z (2013))

Let $I \subset K[x_1, \ldots, x_n]$ be a monomial ideal and $d = t_0(I)$. If $i \geq 0$ and $j \geq i + d$ be non-negative integers such that,

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Corollary (Y-Z (2013))

Let $I \subset S := K[x_1, \ldots, x_n]$ be a non-zero monomial ideal, $\rho = \mathrm{projdim}(I)$ and $d = t_0(I)$. Then,

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(i) $t_{i+1}(S/I) \leq t_i(S/I) + t_1(S/I)$.

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(i) *If* $\beta^K_{i,j}(I) \neq 0$, *then* $i + c \leq j \leq d(i+1)$.

(ii) *If I is square-free monomial ideal and* $\beta^K_{i,j}(I) \neq 0$, *then* $i + c \leq j \leq \min\{n, d(i+1)\}$.

Outline

Definition

If I is a graded ideal of S, then we write $I_{\langle j \rangle}$ for the ideal generated by all homogeneous polynomials of degree j belonging to I. We say that a graded ideal $I \subset S$ is *componentwise linear* if $I_{\langle j \rangle}$ has a linear resolution for all j.

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Let $I \subset S$ be a square-free monomial ideal. For each j, we write $I_{[j]}$ for the ideal generated by all the square-free monomials of degree j belonging to I. We say that I is *square-free componentwise linear*, if $I_{[j]}$ has a linear resolution for all j.

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Proposition (J. Herzog and T. Hibi (1999))

Suppose that $I \subset S$ be a square-free monomials. Then I is componentwise linear if and only if I is square-free componentwise linear.

Theorem (Y-Z (2013))

Let Δ *be a simplicial complex on vertex set* $[n]$, $I := I_\Delta \subset K[x_1, \ldots, x_n]$ *its Stanley-Reisner ideal and* $d = t_0(I)$. *For* $t \geq d$, *let*

$$\Delta_t = \Delta \cup \langle \textit{all } (t-1)\textit{-subsets of } [n] \rangle = \Delta \cup \langle [n] \rangle^{(t-2)}.$$

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Then,

(i) $I_{\Delta_t} = I_{[t]}$;

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Then,

(i) $I_{\Delta_t} = I_{[t]}$;

(ii) *If* $j - i > t$, *then* $\beta^K_{i,j}(I) = \beta^K_{i,j}(I_{[t]})$.

Passing to square-free case

Corollary (Y-Z (2013))

Let $I \subset K[x_1, \ldots, x_n]$ *be a square-free monomial ideal and* $d = t_0(I)$. *Then,*

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Let $I \subset K[x_1, \ldots, x_n]$ be a square-free monomial ideal and $d = t_0(I)$. Then,

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(ii) $\operatorname{reg}(I) = \min\left\{t : \quad t \geq d \text{ and } I_{[t]} \text{ has a } t\text{-linear resolution}\right\}$*;*

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(iii) *If* I *has a* d*-linear resolution, then* $I_{[t]}$ *has a* t*-linear resolution for all* $t \geq d$.

Corollary (Y-Z (2013))

Let $G \neq \mathcal{C}_{n,2}$ *be a graph and* $\mathcal{C}$ *be the* 3*-uniform clutter*

$$\mathcal{C} = \{C : \quad C \text{ is a } 3\text{-cycle in } G\}.$$

Let $I = I(\bar{G}), J = I(\bar{\mathcal{C}})$. *Then,*

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(i) $J = I_{[3]}$.

(ii) J *has a* 3*-linear resolution if and only if* $\operatorname{reg}(I) \leq 3$.

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Let $G \neq \mathcal{C}_{n,2}$ be a graph and $\mathcal{C}$ be the 3-uniform clutter

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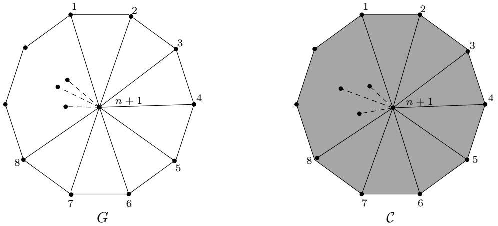
(i) $J = I_{[3]}$.

(ii) *J has a 3-linear resolution if and only if $\operatorname{reg}(I) \leq 3$.*

(iii) *If G is a chordal graph, then the ideal J has a 3-linear resolution.*

Example 1

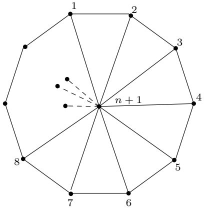
Let G be the the following wheel graph and $\mathcal{C}$ be the 3-uniform clutter consisting of all 3-cycles of G.



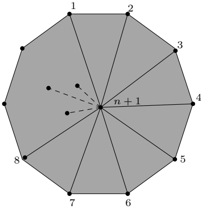
Example 1

Let G be the the following wheel graph and $\mathcal{C}$ be the 3-uniform clutter consisting of all 3-cycles of G. That is:

$$G = C_n \cup \{\{i, n+1\} : \quad 1 \leq i \leq n\},$$
$$\mathcal{C} = \{C : \quad C \text{ is a 3-cycle in } G\}.$$



G



$\mathcal{C}$

Example 1

Note that, $\bar{G} = \bar{C}_n$. Since $\mathrm{reg}\ (I(\bar{G})) = \mathrm{reg}\ (I(\bar{C}_n)) = 3$, the ideal

$$J = I\left(\bar{\mathcal{C}}\right) = I\left(\mathcal{C}_{n+1,3} \setminus \mathcal{C}\right)$$
$$= \left(x_i x_j x_k : \quad \{i,j\} \notin E(G),\ 1 \leq k \leq n+1,\ k \notin \{i,j\}\right)$$

has a 3-linear resolution by the previous Corollary.

Example 2 (Bipyramid)

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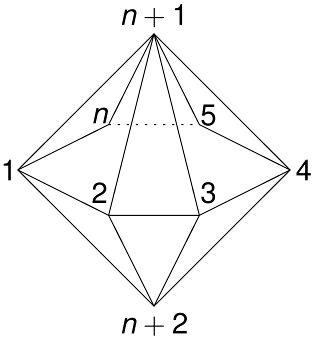

Figure : A Bipyramid $\mathfrak{P}_n$

Corollary (Y-Z (2013))

Let $\mathfrak{P}_n$ *be a bipyramid on* C_n, $n > 3$ *and* G *be the following graph:*

$$G = C_n \cup \{\{i, n+1\} : i \in [n]\} \cup \{\{i, n+2\} : i \in [n]\}.$$

Let $I = I(\bar{G}), J = I(\bar{\mathfrak{P}}_n)$ *be the corresponding circuit ideal in* $S = K[x_1, \ldots, x_{n+2}]$. *Then, the ideal* J *does not have linear resolution and* $\operatorname{reg}(J) = 4$.

For Further Reading I

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Thanks for your attention.