The Annual Seminar on
Mathematical Logic and its Application, December 18 and 19, 2013
Shahid Beheshti University with the cooperation of the School of Mathematics, IPM, Tehran

Cantor's Diagonal Argument: A Characterization

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Cantor's Diagonal Argument came out of his second proof for the uncountability of the set of real numbers. Unlike the first proof, the diagonal argument can also show the non-equinumerosity of a set with its powerset. In modern terms the proof is as follows: for a function $F : A \to \mathscr{P}(A)$, where $\mathscr{P}(A) = \{B \mid B \subseteq A\}$ is the powerset of A, the diagonal set $D_F = \{a \in A \mid a \notin F(a)\}$ is not in the range of F because if it were, say $D_F = F(\alpha)$, then $\alpha \in D_F \leftrightarrow \alpha \notin F(\alpha) \leftrightarrow \alpha \notin D_F$ contradiction. This argument shows up also in Russell's Paradox, the set of sets which do not contain themselves, $R = \{x \mid x \notin x\}$, and in Turing's non-recursively-enumerable set $\overline{K} = \{n \in \mathbb{N} \mid n \notin W_n\}$ where W_n is the domain of the n^{th} recursive function φ_n (i.e., $W_n = \{x \in \mathbb{N} \mid \exists y : \varphi_n(x) = y\}$) by which one can show the algorithmic unsolvability of the halting problem (of a given algorithm on a given input). There are, in fact, many other instances of the diagonal arguments in wide areas of mathematics from logic and set theory to computability theory and theory of computational complexity.

In this talk, we examine this argument in more detail and discuss some other proofs of Cantor's theorem (on the non-equinumerosity of a set with its powerset). By introducing a generalized diagonal argument, we show that all other proofs should fit in this generalized form, which is roughly as follows: for a function $g : A \to A$ the generalized diagonal set $D_F^g = \{g(a) \mid g(a) \notin F(a)\}$ is not in the range of F because if it were, say $D_F^g = F(\alpha)$, then $g(\alpha) \in D_F^g \leftrightarrow g(\alpha) \notin F(\alpha) \leftrightarrow g(\alpha) \notin D_F^g$ contradiction. For the argument to go through, the function g should satisfy some conditions; and we will prove that every subset of A (say $B \subseteq A$) that is not in the range of F (for all $a \in A$, $B \neq F(a)$ holds) should be in this generalized diagonal form ($B = D_F^g$) for some suitable function g which satisfies those conditions. We will argue that this provides a characterization for diagonal proofs and indeed characterizes the objects whose existence are proved by a kind of diagonal(izing out) argument.