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Gorenstein Homology, Relative Pure Homology and Virtually Gorenstein Rings

Fatemeh Zareh-Khoshchehreh
Buein Zahra Technical University

Iran

Let R be a commutative ring with identity and $\mathcal{GP}$ and $\mathcal{GI}$ denote the classes of Gorenstein projective and Gorenstein injective R-modules, respectively. Warfield [2] has introduced a notion of $\mathscr{X}$-purity for any class $\mathscr{X}$ of R-modules. Recall from [2] that an exact sequence $0 \to A \to B \to C \to 0$ of R-modules is called $\mathscr{X}$- *pure exact* if for any $U \in \mathscr{X}$ the induced R-homomorphism $\operatorname{Hom}_R(U, B) \to \operatorname{Hom}_R(U, C)$ is surjective. An R-module M is called $\mathscr{X}$*-pure projective* (respectively $\mathscr{X}$*- pure injective*; $\mathscr{X}$*-pure flat*) if the functor $\operatorname{Hom}_R(M, -)$ (respectively $\operatorname{Hom}_R(-, M)$; $M \otimes_R -$) leaves any $\mathscr{X}$-pure exact sequence exact.

Let $\mathscr{X}$ be a class of R-modules. We say a homology theory $\mathcal{T}$ is a $\mathscr{X}$*-pure homology* if an R-module M is projective (respectively injective; flat) in $\mathcal{T}$ if and only if it is $\mathscr{X}$-pure projective (respectively $\mathscr{X}$-pure injective; $\mathscr{X}$-pure flat). In this talk, we investigate the question: Is Gorenstein homology a $\mathscr{X}$-pure homology for an appropriate class $\mathscr{X}$ of R-modules? Our candidate for a such class $\mathscr{X}$ is $\mathcal{GP}$. To treat this question, we focus on Noetherian rings of finite Krull dimension. So, from now to the end of the abstract, assume that R is Noetherian of finite Krull dimension. It is not hard to verify that

$$\mathcal{GP} = \{M \in R - \operatorname{Mod} \,|\, M \text{ is } \mathcal{GP} - \text{pure projective}\}.$$

On the other hand, we show that if the classes of Gorenstein injectives and $\mathcal{GP}$-pure injectives are the same, then also the classes of Gorenstein flats and $\mathcal{GP}$-pure flats are the same. Therefore, our question reduces to: Are the classes of Gorenstein injectives and $\mathcal{GP}$-pure injectives the same?

For any class $\mathscr{X}$ of R-modules, let $\mathscr{X}^{\perp}$ (respectively ${}^{\perp}\mathscr{X}$) denotes the class of R-modules M with the property that $\operatorname{Ext}^1_R(X, M) = 0$ (respectively $\operatorname{Ext}^1_R(M, X) = 0$) for all R-modules $X \in \mathscr{X}$. we show that ${}^{\perp}\mathcal{GI} \subseteq \mathcal{GP}^{\perp}$. We call R *virtually Gorenstein* if ${}^{\perp}\mathcal{GI} = \mathcal{GP}^{\perp}$. This generalizes the notion of virtually Gorenstein Artin algebras which was introduced by Beligiannis and Reiten in [1]. Our main results make such algebras relevant also in commutative ring theory. We prove that Gorenstein homology is a $\mathcal{GP}$-pure homology if and only if the functor $\mathrm{Hom}_R(-, \sim)$ is right balanced by $\mathcal{GP} \times \mathcal{GI}$ and if and only if R is virtually Gorenstein. This talk is based on a joint work with Mohsen Asgharzadeh and Kamran Divaani-Aazar [3].

References

[1] A. Beligiannis and I. Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc., **188** (883), (2007).

[2] R.B. Warfield, *Purity and algebraic compactness for modules*, Pacific J. Math., **28** (3), (1969), 699-719.

[3] F. Zareh-Khoshchehreh, M. Asgharzadeh and K. Divaani-Aazar, *Gorenstein homology, relative pure homology and virtually Gorenstein rings*, J. Pure Appl. Algebra, **218** (12), (2014), 2356-2366.