Linearity defect, Castelnuovo-Mumford Regularity and Rate of modules

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In this presentation rings are commutative noetherian local (or standard graded algebras over a field), and modules are finitely generated (or graded finitely generated).
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Let \((R, m, k)\) be a local ring (or standard graded \(k\)-algebra), \(M\) an \(R\)-module and \(F \to M\) be the minimal free resolution of \(M\).

1. \(P^R_M(t) = \sum_i \dim_k \Tor^R_i(M, k) t^i\)
2. \(H_M(z) = \sum_{i \in \mathbb{Z}} \dim_k (M_i) z^i \) (\(M\) is graded)
3. \(\reg_R(M) = \sup \{ j - i : \text{and } \Tor^R_i(M, k)_j \neq 0, \ j \in \mathbb{Z}, i \in \mathbb{N} \} \) (\(M\) is graded).
Tor^R(k, k) has a graded algebra structure and Tor^R(M, k) is a graded module over Tor^R(k, k). So

\[ P^R_M(t) = H_{\text{Tor}^R(M,k)}(z) \]
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$$P^R_M(t) = H_{\text{Tor}^R(M, k)}(z)$$
Rationality of $P^R_M(t)$

**Definition**

$P^R_M(t)$ is said to be rational if $P^R_M(t) = f(t)/g(t) \in \mathbb{C}(t)$,

A rational expression for $P^R_M(t)$ has practical applications.

1. it provides recurrent relation for Betti numbers that can be useful in constructing a minimal resolution.
2. it allows for an efficient estimate of asymptotic behavior of Betti sequences.
Question

(Serre, Kaplasky, Shafarevich,...)

Dose $P_k^R(t)$ is a rational function?

D. Anick (1982): Gave a negative answer to the question. $\exists (R, m) s.t. m^3 = 0, \ P_k^R(t)$ is irrational.

J. Tate (1957): If $R$ is a complete intersection, then $P_k^R(t)$ is rational.

C. Jacobsson (1985) shows that the rationality of $P_k^R(t)$ does not imply the rationality of $P_M^R(t)$ for all $R$-module $M$. 
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Problems

(1) Over which $R$ do all modules have rational Poincare series.
(2) Do all rational $P^R_M(t)$ over $R$ have a common denominator?
(3) (1) and (2).

G. Levin (1985) For the problems (1) and (2) it suffices to consider modules of finite length.
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G. Levin (1985) For the problems (1) and (2) it suffices to consider modules of finite length.
By definition $\text{reg}_R(M)$ can be infinite.

**Question**

Over which $R$, $\text{reg}_R(M) < \infty$ for all $R$-module $M$.

The following are equivalent:

(Avramov-Eisenbud-Peeva).

The following conditions are equivalent:

1. $\text{reg}_R(M)$ is finite for every $R$-module $M$;
2. $\text{reg}_R(K)$ is finite;
3. $R$ is Koszul ($\text{reg}_R(K) = 0$).
$R$ is a Koszul algebra $\iff P_k^R(t) = 1/H_R(-t)$ (so $P_k^R(t)$ is rational)
Definition

Let $F$ be the minimal free resolution of $M$. We can construct a complex of graded modules.

$\text{lin}(F) : \cdots \rightarrow F_n^g(-n) \rightarrow F_{n-1}^g(-n+1) : \cdots F_0^g(0) \rightarrow M^g \rightarrow 0$

where $N^g = \text{gr}_m(N)$

$$\text{ld}_R(M) := \sup \{ i \in \mathbb{N}_0 : H_i(\text{lin}(F)) \neq 0 \} \text{ (linearity defect)}.$$

$M$ is said to be Koszul if $\text{ld}_R(M) = 0$. This is equivalent to say that $M^g$ has a linear resolution.
(1) \( \text{Id}_R(M) \leq d \iff \text{Syz}_d(M) \text{ is Koszul} \)

(2) \( \text{Id}_R(M) \leq \infty \implies P_M^R(t) = q(t)/D_R(t) \)
Question

(1) Over which ring $R$ all modules have finite linearity defect.
(2) What is the relation between $\text{ld}_R(M)$ and $\text{reg}_R(M)$

Definition

$R$ is said to be absolutely Koszul if $\text{ld}_R(M)$ for all $R$-module $M$. 

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Definition

$R$ is said to be absolutely Koszul if $\text{ld}_R(M)$ for all $R$-module $M$. 
**Paper 1** Aim: Study the regularity and Koszulness of modules over local rings
Let \((R, m, k)\) be a local ring which has an element \(x \neq 0 = x^2\) and \(xm = m^2\). Then \(R\) is absolutely Koszul (Avramov- Iyengar-Sega).

1. \(M\) is Koszul if \(xM = 0\);
2. \(\text{reg}_R(M) \leq 1\);
3. \(mM\) is Koszul;
4. For every ideal \(I\) of \(R\) the quotient ring \(R/I\) is a Koszul ring.
Let \((R, \mathfrak{m}, k)\) be a Cohen-Macaulay local ring of dimension \(d > 0\) and of minimal multiplicity. Then any \(R\)-module annihilated by a minimal reduction of \(\mathfrak{m}\) is a Koszul module. In particular, \(R\) is a Koszul ring.
Let \((R, \mathfrak{m}, k)\) be a Cohen-Macaulay local ring of dimension \(d > 0\) and of minimal multiplicity. Then any \(R\)-module annihilated by a minimal reduction of \(\mathfrak{m}\) is a Koszul module. In particular, \(R\) is a Koszul ring.
Let \((R, \mathfrak{m})\) be a local ring and \(M\) be an \(R\)-module. If \(\text{Ann}(x)\mathfrak{m} = \mathfrak{m}^2\), for all \(x \in \mathfrak{m} \setminus \mathfrak{m}^2\), then the following hold.

(i) \(M\) is Koszul if there exists \(x \in \mathfrak{m} \setminus \mathfrak{m}^2\) such that \(\text{Ann}(x)M = 0\); in particular, \(R\) is a Koszul ring.

(ii) If \(x_1, \cdots, x_s\) are in \(\mathfrak{m} \setminus \mathfrak{m}^2\), then \((x_1, \cdots, x_s)M\) is Koszul;

(iii) \(\text{reg}_R(M) \leq 1\);

(iv) for every ideal \(I \subseteq \mathfrak{m}\) the quotient ring \(R/I\) is Koszul;
Paper 2 (Joint with Rossi) Aim: study the relation between $\text{Id}_R(M)$ and $\text{reg}_R(M)$ and answer to a question asked by Herezog and Iyengar.
Let $R$ be a standard graded algebra and let $M$ be a graded $R$-module with $D(M) = \{i_1, \cdots, i_s\}$. Assume that $M$ is Koszul. Then for each $n \geq 1$ we have

$$\text{Tor}_n^R(M, k)_j = 0 \quad \text{for } j \neq i_1 + n, \cdots, i_s + n.$$ 

Furthermore

if for some $n \geq 1$ and $1 \leq r \leq s$, we have $\text{Tor}_n^R(M, k)_{i_r+n} = 0$, then $\text{Tor}_m^R(M, k)_{i_r+m} = 0$, for all $m \geq n$. 

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$$\text{Tor}^R_n(M, k)_j = 0 \quad \text{for} \quad j \neq i_1 + n, \cdots, i_s + n.$$  

Furthermore if for some $n \geq 1$ and $1 \leq r \leq s$, we have $\text{Tor}^R_n(M, k)_{i_r + n} = 0$, then $\text{Tor}^R_m(M, k)_{i_r + m} = 0$, for all $m \geq n$. 

Let $R$ be a standard graded algebra and let $M$ be a $R$-module. If $\text{Id}_R(M) = d < \infty$ then

$$\text{reg}_R(M) = \max\{t_i(M) - i : 0 \leq i \leq d\}.$$ 

In particular, if $M$ is Koszul then $\text{reg}_R(M) = t_0(M)$. 
Let $R$ be a standard graded $k$-algebra. If $\text{ld}(k) < \infty$ then $\text{ld}(k) = 0$. 

(Herzog, Iyengar)
Question

Herzog, Iyengar (2005) Let \((R, \mathfrak{m}, k)\) be a local ring. If \(\text{ld}(k) < \infty\) then is \(\text{ld}(k) = 0?\).
positive Answers

(i) (Sega 2013) $m^3 = 0$

(ii) $R$ is complete intersection and $R^g$ is CM

(iii) (Rossi, - (2013)) $R$ is of homogeneous type.
Definition

Backelin Let $R$ be a standard graded $k$-algebra

$$\text{Rate}(R) := \sup\{(t_i^R(K) - 1)/i - 1 : \ i \geq 2\}$$

Herzog, Aramova:

$$\text{rate}_R(M) := \sup\{t_i^R(M)/i : \ i \geq 1\}.$$  

A comparison with Bakelin’s rate shows that  
$\text{Rate}(R) = \text{rate}_R(m(1))$. The rate of any module is finite.
Backelin

for all \( d \geq 1 \)

\[
\text{Rate } R^{(d)} \leq \lceil \text{Rate}(R)/d \rceil
\]

Eisenbud, Reeve and Totaro started their work (initial ideals, veronese subrings, ...) from a request by George Kempf for a simpler proof. They showed for \( d \gg 0 \) \( R^{(d)} \) admits a quadratic initial ideals.
for all $d \geq 1$

\[
\text{rate}_{R(d)}(M^{(d)}) \leq \max \{ \lceil \text{rate}_R(M)/d \rceil, \lceil t_0^R(M)/d \rceil \}
\]

A direct consequence of this is Backelin’s Theorem. If $d \geq \text{rate}_R(M)$, then $M^{(d)}$ is a Koszul module over $R^{(d)}$. 

Jahangiri,-
Thanks.