Linearity defect, Castelnuovo-Mumford Regularity and Rate of modules

Rasoul Ahangari Maleki

June 18, 2014

tu-logo

ur-logo

Contents

tu-logo

ur-logo

Contents

In this presentation rings are commutative noetherian local (or standard graded algebras over a field), and modules are finitely generated (or graded finitely generated).

In this presentation rings are commutative noetherian local (or standard graded algebras over a field), and modules are finitely generated (or graded finitely generated).

Let $(R, \mathfrak{m}, k)$ be a local ring (or standard graded k-algebra), M an R-module and $\mathbf{F} \to M$ be the minimal free resolution of M.

(1) $\mathrm{P}_M^R(t) = \sum_i \dim_k \mathrm{Tor}_i^R(M, k) t^i$

(2) $\mathrm{H}_M(z) = \sum_{i \in \mathbb{Z}} \dim_k (M_i) z^i$ (M is graded)

(3) $\mathrm{reg}_R(M) = \sup\{j - i : \text{and } \mathrm{Tor}_i^R(M, k)_j \neq 0, \ \ j \in \mathbb{Z}, i \in \mathbb{N}\}$ (M is graded).

$\mathrm{Tor}^R(k,k)$ has a graded algebra structure and $\mathrm{Tor}^R(M,k)$ is a graded module over $\mathrm{Tor}^R(k,k)$. So

$$\mathrm{P}_M^R(t) = \mathrm{H}_{\mathrm{Tor}^R(M,k)}(z)$$

tu-logo

ur-logo

$\operatorname{Tor}^R(k,k)$ has a graded algebra structure and $\operatorname{Tor}^R(M,k)$ is a graded module over $\operatorname{Tor}^R(k,k)$. So

$$\mathrm{P}_M^R(t) = \mathrm{H}_{\operatorname{Tor}^R(M,k)}(z)$$

tu-logo

ur-logo

Rationality of $\mathrm{P}_M^R(t)$

Definition

$\mathrm{P}_M^R(t)$ *is said to be rational if* $\mathrm{P}_M^R(t) = f(t)/g(t) \in \mathbb{C}(t)$,

A rational expression for $\mathrm{P}_M^R(t)$ has practical applications.

(1) it provides recurrent relation for Betti numbers that can be useful in constructing a minimal resolution.

(2) it allows for a efficient estimate of asymptotic behavior of Betti sequences.

tu-logo

ur-logo

Question

(Serre, Kaplasky, Shafarevich,...)
Dose $P_k^R(t)$ is a rational function?

D. Anick (1982): Gave a negative answer to the question. $\exists(R,\mathfrak{m}) s.t\ \ \mathfrak{m}^3=0,\ \ P_k^R(t)$ is irrational.
J. Tate (1957): If R is a complete intersection, then $P_k^R(t)$ is rational.
C. Jacobsson (1985) shows that the rationality of $P_k^R(t)$ dose not imply the rationality of $P_M^R(t)$ for all R-module M.

tu-logo

ur-logo

Question

(Serre, Kaplasky, Shafarevich,...)
Dose $\mathrm{P}_k^R(t)$ is a rational function?

D. Anick (1982): Gave a negative answer to the question.
$\exists(R, \mathfrak{m}) s.t\ \ \mathfrak{m}^3 = 0, \ \ \mathrm{P}_k^R(t)$ is irrational.
J. Tate (1957): If R is a complete intersection, then $\mathrm{P}_k^R(t)$ is rational.
C. Jacobsson (1985) shows that the rationality of $\mathrm{P}_k^R(t)$ dose not imply the rationality of $\mathrm{P}_M^R(t)$ for all R-module M.

tu-logo

ur-logo

However counterexamples do not seem abound.

However counterexamples do not seem abound.

Problems

(1) Over which R do all modules have rational poincare series.
(2) Do all rational $P_M^R(t)$ over R have a common denominator?
(3) (1) and (2).

G. Levin (1985) For the problems (1) and (2) it suffices to consider modules of finite length.

tu-logo

ur-logo

Problems

(1) Over which R do all modules have rational poincare series.

(2) Do all rational $\mathrm{P}_M^R(t)$ over R have a common denominator?

(3) (1) and (2).

G. Levin (1985) For the problems (1) and (2) it suffices to consider modules of finite length.

Castelnuovo-Mumford Regularity

By definition $\operatorname{reg}_R(M)$ can be infinite.

Question

Over which R, $\operatorname{reg}_R(M) < \infty$ for all R-module M.

The following are equivalent:

(Avramov-Eisenbud-Peeva).
The following conditions are equivalent:

(1) $\operatorname{reg}_R(M)$ is finite for every R-module M;

(2) $\operatorname{reg}_R(K)$ is finite;

(3) R is Koszul ($\operatorname{reg}_R(K) = 0$).

R is a Koszul algebra $\Leftrightarrow \mathrm{P}_k^R(t) = 1/H_R(-t)$ (so $\mathrm{P}_k^R(t)$ is rational)

tu-logo

ur-logo

Definition

Let **F** *be the minimal free resolution of* M*. We can construct a complex of graded modules.*

$$\mathrm{lin}(\mathbf{F}) : \cdots \to F_n^g(-n) \to F_{n-1}^g(-n+1) :\to \cdots F_0^g(0) \to M^g \to 0$$

where $N^g = gr_{\mathfrak{m}}(N)$

$$\mathrm{ld}_R(M) := \sup\{i \in \mathbb{N}_0 : H_i(\mathrm{lin}(\mathbf{F})) \neq 0\}(\textit{linearity defect}).$$

M is said to be Koszul if $\mathrm{ld}_R(M) = 0$*. This is equivalent to say that* M^g *has a linear resolution.*

(1) $\mathrm{ld}_R(M) \leq d \Leftrightarrow Syz_d(M)$ is Koszul

(2) $\mathrm{ld}_R(M) \leq \infty \rightarrow \mathrm{P}_M^R(t) = q(t)/D_R(t)$

tu-logo

ur-logo

Question

(1) Over which ring R all modules have finite linearity defect.

(2) What is the relation between $\operatorname{ld}_R(M)$ and $\operatorname{reg}_R(M)$

Definition

R is said to be absolutely Koszul if $\operatorname{ld}_R(M)$ for all R-module M.

tu-logo

ur-logo

Question

(1) Over which ring R all modules have finite linearity defect.

(2) What is the relation between $\operatorname{ld}_R(M)$ and $\operatorname{reg}_R(M)$

Definition

R is said to be absolutely Koszul if $\operatorname{ld}_R(M)$ for all R-module M.

tu-logo

ur-logo

Paper 1 Aim: Study the regularity and Koszulness of modules over local rings

Let $(R, \mathfrak{m}, k)$ be a local ring which has an element $x \neq 0 = x^2$ and $x\mathfrak{m} = \mathfrak{m}^2$. Then R is absolutely Koszul (Avramov- Iyengar-Sega)

1. M is Koszul if $xM = 0$;
2. $\operatorname{reg}_R(M) \leq 1$;
3. $\mathfrak{m}M$ is Koszul;
4. For every ideal I of R the quotient ring R/I is a Koszul ring.

tu-logo

ur-logo

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension $d > 0$ and of minimal multiplicity. Then any R-module annihilated by a minimal reduction of $\mathfrak{m}$ is a Koszul module. In particular, R is a Koszul ring.

tu-logo

ur-logo

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension $d > 0$ and of minimal multiplicity. Then any R-module annihilated by a minimal reduction of $\mathfrak{m}$ is a Koszul module. In particular, R is a Koszul ring.

Let $(R, \mathfrak{m})$ be a local ring and M be an R-module. If $\mathrm{Ann}(x)\mathfrak{m} = \mathfrak{m}^2$, for all $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, then the following hold.

(*i*) M is Koszul if there exists $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\mathrm{Ann}(x)M{=}0$; in particular, R is a Koszul ring.

(*ii*) If $x_1, \cdots, x_s$ are in $\mathfrak{m} \setminus \mathfrak{m}^2$, then $(x_1, \cdots, x_s)M$ is Koszul;

(*iii*) $\mathrm{reg}_R(M) \leq 1$;

(*iv*) for every ideal $I \subseteq \mathfrak{m}$ the quotient ring R/I is Koszul;

tu-logo

ur-logo

Paper 2 (Joint with Rossi) Aim: study the relation between $\operatorname{ld}_R(M)$ and $\operatorname{reg}_R(M)$ and answer to an question asked by Herezog and Iyengar.

(Rossi,-)

Let R be a standard graded algebra and let M be a graded R-module with $\mathrm{D}(M) = \{i_1, \cdots, i_s\}$. Assume that M is Koszul. Then for each $n \geq 1$ we have

$$\mathrm{Tor}_n^R(M,k)_j = 0 \quad \textit{for } j \neq i_1 + n, \cdots, i_s + n.$$

Furthermore

if for some $n \geq 1$ and $1 \leq r \leq s$, we have $\mathrm{Tor}_n^R(M,k)_{i_r+n} = 0$, then $\mathrm{Tor}_m^R(M,k)_{i_r+m} = 0$, for all $m \geq n$.

tu-logo

ur-logo

(Rossi,-)

Let R be a standard graded algebra and let M be a graded R-module with $\mathrm{D}(M) = \{i_1, \cdots, i_s\}$. Assume that M is Koszul. Then for each $n \geq 1$ we have

$$\mathrm{Tor}_n^R(M,k)_j = 0 \quad \textit{for } j \neq i_1 + n, \cdots, i_s + n.$$

Furthermore
if for some $n \geq 1$ and $1 \leq r \leq s$, we have $\mathrm{Tor}_n^R(M,k)_{i_r+n} = 0$, then $\mathrm{Tor}_m^R(M,k)_{i_r+m} = 0$, for all $m \geq n$.

(Rossi,-)

Let R be a standard graded algebra and let M be a R-module. If $\mathrm{ld}_R(M) = d < \infty$ then

$$\mathrm{reg}_R(M) = \max\{t_i(M) - i : 0 \leq i \leq d\}.$$

In particular, if M is Koszul then $\mathrm{reg}_R(M) = t_0(M)$.

tu-logo

ur-logo

(Herzog, Iyengar)

Let R be a standard graded k-algebra. If $\operatorname{ld}(k) < \infty$ then $\operatorname{ld}(k) = 0$.

Question

Herzog, Iyengar (2005) Let $(R, \mathfrak{m}, k)$ *be a a local ring. If* $\operatorname{ld}(k) < \infty$ *then is* $\operatorname{ld}(k) = 0$*?.*

positive Answers

(*i*) (Sega 2013) $\mathfrak{m}^3 = 0$

(*ii*) R is complete intersection and R^g is CM

(*iii*) (Rossi,- (2013)) R is of homogeneous type.

Definition

Backelin Let R be a standard graded k-algebra

$$\mathrm{Rate}(R) := \sup\{(t_i^R(K) - 1)/i - 1 : \quad i \geq 2\}$$

Herzog , Aramova:

$$\mathrm{rate}_R(M) := \sup\{t_i^R(M)/i : \quad i \geq 1\}.$$

A comparison with Bakelin's rate shows that $\mathrm{Rate}(R) = \mathrm{rate}_R(\mathfrak{m}(1))$*. The rate of any module is finite.*

Backelin

for all $d \geq 1$

$$\text{Rate } R^{(d)} \leq \lceil \text{Rate}(R)/d \rceil$$

Eisenbud, Reevse and Totaro started their work (initial ideals, veronese subrings,...) from a request by George Kempf for a simpler proof. They showed for $d >> 0$ $R^(d)$ admits a quadratic initial ideals.

tu-logo

ur-logo

Jahangiri,-

for all $d \geq 1$

$$\mathrm{rate}_{R^{(d)}}(M^{(d)}) \leq \max\{\lceil \mathrm{rate}_R(M)/d \rceil, \lceil t_0^R(M)/d \rceil\}$$

A direct consequence of this is Backelin's Theorem.
If $d \geq \mathrm{rate}_R(M)$, then $M^{(d)}$ is a Koszul module over $R^{(d)}$.

tu-logo

ur-logo

Thanks .