Auslander class and C−projective modules modulo exact zero-divisors

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11th seminar on commutative algebra and related topics
School of Mathemaics, IPM, 2014

Exact zero-divisors

Throughout, R is a commutative and noetherian ring and M an R–module.

Exact zero–divisors introduced by Henriques and Şega, in 2009.

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Definition

An element x of R is called an *exact zero–divisor* on M if $xM \neq 0$, $xM \neq M$ and there is $y \in R$ such that the sequence of multiplication maps $M \xrightarrow{x} M \xrightarrow{y} M \xrightarrow{x} M$ is exact.
In this case we say that x, y form a *pair* of exact zero–divisors on M.

Example

Let $a \in R$ be an R–regular element. For some integer $r > 1$, the sequence of multiplication maps

$$R/(a^r) \xrightarrow{a^s} R/(a^r) \xrightarrow{a^{r-s}} R/(a^r) \xrightarrow{a^s} R/(a^r)$$

is exact for all $0 < s < r$.
Then a^s, a^{r-s} form a pair of exact zero–divisors on $R/(a^r)$.

Exact zero-divisors

Basic properties

Let R be a local ring, x an exact zero–divisor on R.

- $\dim(R) = \dim(R/xR)$ and $\mathrm{depth}(R) = \mathrm{depth}(R/xR)$. (Avramov, Henriques and Şega, in 2010)
- R is Cohen-Macaulay if and only if R/xR is Cohen-Macaulay. (Avramov, Henriques and Şega, in 2010)
- R is Gorenstein if and only if R/xR is Gorenstein ring. (Henriques and Şega, in 2009)
- $\mathrm{pd}_R(R/xR) = \infty$.

Exact zero-divisors

Proposition (Dibaei and Gheibi)

Assume that x, y form a pair of exact zero–divisors on R. Let n be a non-negative integer. Consider the following statements.

(i) x, y form a pair of exact zero–divisors on M.

(ii) $\mathrm{Ext}^i_R(R/xR, M) = 0$ for all $i > n$.

(iii) $\mathrm{Tor}_i^R(R/xR, M) = 0$ for all $i > n$.

Then (i)⇒(ii)⇔(iii). If one of the following conditions holds true, then the statements (i), (ii) and (iii) are equivalent.

(a) $xM \neq 0$ and $xM \neq M$.

(b) R is local and M is finite.

Exact zero-divisors

We observed that, if x, y form a pair of exact zero–divisors on R and if $M \xrightarrow{x} M$ is neither zero nor epimorphism, then x, y form also a pair of exact zero–divisors on M whenever one of the conditions $\mathrm{id}(M) < \infty$, $\mathrm{pd}(M) < \infty$, or $\mathrm{fd}(M) < \infty$ holds true.

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Proposition (Dibaei and Gheibi)

Assume that x, y form a pair of exact zero–divisors on both R and M and that N is an R/xR–module. Then the following statements hold true for all $i \geqslant 0$.

(i) $\mathrm{Ext}^i_R(N, M) \cong \mathrm{Ext}^i_{R/xR}(N, M/xM)$.

(ii) $\mathrm{Ext}^i_R(M, N) \cong \mathrm{Ext}^i_{R/xR}(M/xM, N)$.

(iii) $\mathrm{Tor}^R_i(M, N) \cong \mathrm{Tor}^{R/xR}_i(M/xM, N)$.

Semidualizing modules

Definition

An R–module C is called *semidualizing*, if

- C is finitely generated
- The natural homothety map $\chi_C^R : R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism
- For all $i > 0$, $\mathrm{Ext}^i_R(C, C) = 0$

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Example

Examples of semidualizing modules include

- R
- The dualizing module of R if it exists (dualizing module is a semidualizing module with finite injective dimension).

These are called the trivial semidualizing modules.

Semidualizing modules

Throughout C assumed to be a semidualizing R–module.

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Basic properties

- $\operatorname{Ann}_R(C) = 0$ and $\operatorname{Supp}_R(C) = \operatorname{Spec}(R)$.
- $\dim_R(C) = \dim(R)$ and $\operatorname{Ass}_R(C) = \operatorname{Ass}_R(R)$.
- An element $a \in R$ is R–regular if and only if it is C–regular.
- If $a \in R$ is R–regular, then C/aC is semidualizing R/aR–module.
- If R is local, then $\operatorname{depth}_R(C) = \operatorname{depth}(R)$.

Semidualizing and exact zero-divisor

Note that if $x \in R$ is non–zero, then $xC \neq 0$. By Nakayama's lemma, $xC = C$ if and only if $(x) = R$.
Let x, y form a pair of exact zero–divisors on R.

- If $\mathrm{pd}(C) < \infty$, then Auslander-Buchsbaum formula implies that C is projective and so x, y form a pair of exact zero–divisors on C by definition.
- If R is Cohen-Macaulay local ring with dualizing module ω, then x, y form also a pair of exact zero–divisors on ω.

In general, we do not know whether a pair of exact zero–divisors on R is also a pair of exact zero–divisors on C.

Semidualizing and exact zero-divisor

Proposition (Dibaei, me)

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring that is not Gorenstein, with dualizing module ω. Let $f : R \longrightarrow S$ be a flat local ring homomorphism such that $S/\mathfrak{m}S$ is not Gorenstein. Assume that $x, y \in S$ form a pair of exact zero–divisors on S such that $\mathrm{fd}_R(S/xS) < \infty$. Then $S \otimes_R \omega$ is a semidualizing S–module which is not a dualizing S–module and $\mathrm{pd}_S(S \otimes_R \omega) = \infty$. Moreover, x, y form a pair of exact zero–divisors on $S \otimes_R \omega$.

Semidualizing and exact zero-divisor

Example (Dibaei, me)

Let $R = k[X, Y]/(X, Y)^2$, whenever k is a field. Then R is a local artinian ring that is not Gorenstein. As R is free k–module of rank 3, $\omega = \mathrm{Hom}_k(R, k)$ is dualizing R–module. Set $S = R[U, V, W, Z]/(U^2, VW, VZ)$. Then S is free R–module and $S/\mathfrak{m}S \cong k[U, V, W, Z]/(U^2, VW, VZ)$ is not Cohen-Macaulay, where $\mathfrak{m}$ is the maximal ideal of R. If u is the image of U in S, then u, u form a pair of exact zero–divisors on S. We have an R–isomorphism $S/uS \cong R[V, W, Z]/(VW, VZ)$ and so S/uS is free R–module. Thus u, u form also a pair of exact zero–divisors on the semidualizing S–module $S \otimes_R \omega$.
Note that $S \otimes_R \omega$ is not a dualizing S–module with $\mathrm{pd}_S(S \otimes_R \omega) = \infty$.

Semidualizing R/xR*–modules*

Proposition (Dibaei, me)

Let B be a finite R–module. Assume that x, y form a pair of exact zero–divisors on both R and B. Then the following statements are equivalent.

(i) B is a semidualizing R–module.

(ii) B/xB and B/yB are semidualizing R/xR– and R/yR–modules, respectively.

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Corollary (Dibaei, me)

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring, D a finite R–module. Assume that x, y form a pair of exact zero–divisors on both R and D. If D/xD is a dualizing R/xR–module and D/yD is a semidualizing R/yR–module, then D is a dualizing R–module.

Note that the converse of the corollary was proved by Dibaei and Gheibi, in 2011.

Definition

The class $\mathcal{G}_C(R)$ consists of G_C–*projective* R–modules, i.e. the class of all finite R–modules M which satisfy the following conditions.

- The natural homomorphism $\delta_M^C : M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism.
- For all $i > 0$, $\mathrm{Ext}^i_R(M, C) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C)$.

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It is easy to see that if x, y form a pair of exact zero–divisors on R, then $R/xR \in \mathcal{G}_R(R)$.

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Proposition (Dibaei, me)

If x, y form a pair of exact zero–divisors on both R and C, then

$$R/xR \in \mathcal{G}_C(R).$$

Auslander class and Bass class

Definition

The *Auslander class* $\mathcal{A}_C(R)$ with respect to C is the class of all R–modules M satisfying the following conditions.

- The natural map $\gamma_M^C : M \longrightarrow \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.
- For all $i > 0$, $\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_R M)$.

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- For all $i > 0$, $\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_R M)$.

Definition

The *Bass class* $\mathcal{B}_C(R)$ with respect to C is the class of all R–modules M satisfying the following conditions.

- The evaluation map $\xi_M^C : C \otimes_R \mathrm{Hom}_R(C, M) \longrightarrow M$ is an isomorphism.
- For all $i > 0$, $\mathrm{Ext}_R^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, M))$.

The classes $\mathcal{A}_C(R)$, $\mathcal{B}_C(R)$ *and* $\mathcal{G}_C(R)$

Proposition (Dibaei, me)

If x, y form a pair of exact zero–divisors on both R and C, then

$$R/xR \in \mathcal{A}_C(R).$$

The classes $\mathcal{A}_C(R)$, $\mathcal{B}_C(R)$ and $\mathcal{G}_C(R)$

Proposition (Dibaei, me)

If x, y form a pair of exact zero–divisors on both R and C, then

$$R/xR \in \mathcal{A}_C(R).$$

Corollary (Dibaei, me)

Assume that x, y form a pair of exact zero–divisors on both R and C. If T is an R/xR–module, then the following statements hold true.

(i) If T is finite, then $T \in \mathcal{G}_C(R)$ if and only if $T \in \mathcal{G}_{C/xC}(R/xR)$.

(ii) $T \in \mathcal{A}_C(R)$ if and only if $T \in \mathcal{A}_{C/xC}(R/xR)$.

(iii) $T \in \mathcal{B}_C(R)$ if and only if $T \in \mathcal{B}_{C/xC}(R/xR)$.

Proposition (Dibaei, me)

Assume that x, y form a pair of exact zero–divisors on R, C and M.

- If $M/xM \in \mathcal{A}_{C/xC}(R/xR)$ and $M/yM \in \mathcal{A}_{C/yC}(R/yR)$, then $M \in \mathcal{A}_C(R)$.
- If $M/xM \in \mathcal{B}_{C/xC}(R/xR)$ and $M/yM \in \mathcal{B}_{C/yC}(R/yR)$, then $M \in \mathcal{B}_C(R)$.
- If M is finite, $M/xM \in \mathcal{G}_{C/xC}(R/xR)$ and $M/yM \in \mathcal{G}_{C/yC}(R/yR)$, then $M \in \mathcal{G}_C(R)$.

Question (Dibaei, me)

Assume that x, y form a pair of exact zero–divisors on both R and M.

- If $M \in \mathcal{G}_C(R)$, is $M/xM \in \mathcal{G}_C(R)$?
- If $M \in \mathcal{B}_C(R)$, is $M/xM \in \mathcal{B}_C(R)$?
- If $M \in \mathcal{A}_C(R)$, is $M/xM \in \mathcal{A}_C(R)$?

The classes $\mathcal{A}_C(R)$, $\mathcal{B}_C(R)$ and $\mathcal{G}_C(R)$

Proposition (Dibaei, me)

Assume that x, y form a pair of exact zero–divisors on both R and M.

- If $M \in \mathcal{G}_C(R)$, then $M/xM \in \mathcal{G}_C(R)$ if and only if x, y form also a pair of exact zero–divisors on $\mathrm{Hom}_R(M, C)$.
- If $M \in \mathcal{B}_C(R)$, then $M/xM \in \mathcal{B}_C(R)$ if and only if x, y form also a pair of exact zero–divisors on $\mathrm{Hom}_R(C, M)$.
- If $M \in \mathcal{A}_C(R)$ is finite , then $M/xM \in \mathcal{A}_C(R)$ if and only if x, y form also a pair of exact zero–divisors on $C \otimes_R M$.

The classes $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$

Definition

The classes of *C–injective*, *C–projective* and *C–flat* modules are defined, respectively, as

$$\mathcal{I}_C(R) = \{ \operatorname{Hom}_R(C, I) \mid I \text{ is an injective } R\text{–module}\},$$
$$\mathcal{P}_C(R) = \{ C \otimes_R P \mid P \text{ is a projective } R\text{–module}\},$$
$$\mathcal{F}_C(R) = \{ C \otimes_R F \mid F \text{ is a flat } R\text{–module}\}.$$

They are the classes of injective, projective and flat R–modules, respectively, when $C = R$.

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They are the classes of injective, projective and flat R–modules, respectively, when $C = R$.

For any R–module M there exists an *augmented proper $\mathcal{P}_C$–projective resolution*, that is, a complex

$$X^+ = \cdots \xrightarrow{\partial_2^X} C \otimes_R P_1 \xrightarrow{\partial_1^X} C \otimes_R P_0 \xrightarrow{\partial_0^X} M \longrightarrow 0$$

such that $\operatorname{Hom}_R(C \otimes_R Q, X^+)$ is exact for all projective R–module Q.

The truncated complex

$$X = \cdots \xrightarrow{\partial_2^X} C \otimes_R P_1 \xrightarrow{\partial_1^X} C \otimes_R P_0 \longrightarrow 0$$

is called a *proper* $\mathcal{P}_C$*–projective resolution* of M.
An *augmented proper* $\mathcal{F}_C$*–projective resolution* for M is defined similarly.
Dually, for any R–module N there exists an *augmented proper* $\mathcal{I}_C$*–injective resolution*, that is, a complex

$$Y^+ = 0 \longrightarrow N \longrightarrow \mathrm{Hom}_R(C, I^0) \xrightarrow{\partial_Y^0} \mathrm{Hom}_R(C, I^1) \xrightarrow{\partial_Y^1} \cdots$$

such that $\mathrm{Hom}_R(Y^+, \mathrm{Hom}_R(C, I))$ is exact for all injective R–module I.

$\mathcal{P}_C$–*projective dimension*

Definition

The $\mathcal{P}_C$–*projective dimension* of an R–module M is

$$\mathcal{P}_C - \mathrm{pd}(M) = \inf \left\{ \sup X \;\middle|\; \begin{array}{c} X \text{ is a proper } \mathcal{P}_C - \text{projective} \\ \text{resolution of } M \end{array} \right\}$$

where $\sup X = \sup\{n \mid X_n \neq 0\}$. The modules of zero $\mathcal{P}_C$–projective dimensions are the non-zero modules in $\mathcal{P}_C(R)$; and we set $\mathcal{P}_C - \mathrm{pd}(0) = -\infty$.

The $\mathcal{F}_C$–*projective dimension*, denoted $\mathcal{F}_C - \mathrm{pd}(-)$, is defined similarly and the $\mathcal{I}_C$–*injective dimension*, denoted $\mathcal{I}_C - \mathrm{id}(-)$, is defined dually.

The class $\mathcal{P}_C(R)$ modulo exact zero-divisors

Proposition (Dibaei, me)

Let x, y form a pair of exact zero–divisors on both R and C. Assume that M is either in $\mathcal{I}_C(R)$, $\mathcal{P}_C(R)$, or $\mathcal{F}_C(R)$ such that $M \xrightarrow{x} M$ is neither zero nor epimorphism. Then x, y form a pair of exact zero-divisors on M.
In any such case M/xM belongs to $\mathcal{I}_{C/xC}(R/xR)$, $\mathcal{P}_{C/xC}(R/xR)$, or $\mathcal{F}_{C/xC}(R/xR)$, respectively.

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Proposition (Dibaei, me)

Assume that x, y form a pair of exact zero–divisors on both R and C. If $\mathcal{P}_C\text{–pd}(M)$, $\mathcal{F}_C\text{–pd}(M)$, or $\mathcal{I}_C\text{–id}(M)$ is finit with $xM \neq 0$ and $xM \neq M$, then x, y form a pair of exact zero–divisors on M.

$\mathcal{P}_C$–projective dimension modulo exact zero-divisors

Proposition (Dibaei, me)

Assume that x, y form a pair of exact zero–divisors on both R and C. Let $xM \neq 0$ and $xM \neq M$ and set $\overline{(-)} = (-) \otimes_R R/xR$. The following statements hold true.

(i) If $\mathcal{P}_C - \mathrm{pd}(M) < \infty$, then $\mathcal{P}_{\overline{C}} - \mathrm{pd}(\overline{M}) \leqslant \mathcal{P}_C - \mathrm{pd}(M)$.

(ii) If $\mathcal{F}_C - \mathrm{pd}(M) < \infty$, then $\mathcal{F}_{\overline{C}} - \mathrm{pd}(\overline{M}) \leqslant \mathcal{F}_C - \mathrm{pd}(M)$.

(iii) If M is finite with $\mathcal{I}_C - \mathrm{id}(M) < \infty$, then $\mathcal{I}_{\overline{C}} - \mathrm{id}(\overline{M}) \leqslant \mathcal{I}_C - \mathrm{id}(M)$.

(iv) If R is local and M is finite, then equality holds in (i), (ii) and (iii).

Thank You