Moore's conjecture for (strongly) Gorenstein flat modules

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Definitions

Let Γ be a group, R be a commutative ring with identity and $R\Gamma$ be its associated group ring. A (left) $R\Gamma$-module is an R-module A together with an action of Γ on A. For example, one has for any R-module A, the *trivial* module structure, with $ga = a$ for $g \in \Gamma$ and $a \in A$.

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- If M is a projective $R\Gamma$-module, then it is projective as an $R\Gamma'$-module.
- (**Question.**) **Is the converse of the second assertion true?**

Counterexample

(**Example:**) Assume that Γ is a non-trivial finite group and take $R = \mathbb{Z}$, Γ' is a trivial subgroup of Γ and $M \cong \mathbb{Z}$ with the trivial Γ-action. So, M is free over $\mathbb{Z}\Gamma'$ but it is not projective over $\mathbb{Z}\Gamma$.

Let Γ, Γ' and R be as stated above. Assume that the index Γ' in Γ is invertible in R. Then the $R\Gamma$-homomorphism $R\Gamma \otimes_{R\Gamma'} M \longrightarrow M$ splits. So, M being projective over $R\Gamma'$ implies that it is also projective as an $R\Gamma$-module as well.

Moore's Conjecture, 1976

Let Γ be a group and Γ' be a subgroup of Γ of finite index and R be any ring with identity. Assume that for all $x \in (\Gamma - \Gamma')$, at least one of the following two conditions holds:

- there is an integer n, such that $1 \neq x^n \in \Gamma'$.

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Let Γ be a group and Γ' be a subgroup of Γ of finite index and R be any ring with identity. Assume that for all $x \in (\Gamma - \Gamma')$, at least one of the following two conditions holds:

- there is an integer n, such that $1 \neq x^n \in \Gamma'$.
- the order of x is finite and invertible in R.

Then every $R\Gamma$-module M which is projective over $R\Gamma'$ is projective over $R\Gamma$ as well.

remarks

- Moore's conjecture may be rephrased as follows; Let Γ, Γ' and R be as above, then for any $R\Gamma$-module M, $\operatorname{pd}_{R\Gamma} M = \operatorname{pd}_{R\Gamma'} M$.

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- Moore's conjecture may be rephrased as follows; Let Γ, Γ' and R be as above, then for any $R\Gamma$-module M, $\operatorname{pd}_{R\Gamma} M = \operatorname{pd}_{R\Gamma'} M$.
- It is easy to see that if $\operatorname{pd}_{R\Gamma} M < \infty$, then $\operatorname{pd}_{R\Gamma} M = \operatorname{pd}_{R\Gamma'} M$. So the real content of Moore's conjecture is the statement: If the triple (Γ, Γ', R) satisfies Moore's condition, then $\operatorname{pd}_{R\Gamma'} M < \infty \Rightarrow \operatorname{pd}_{R\Gamma} M < \infty$.

remarks

- If $R = \mathbb{Z}$ the ring of integers and Γ is torsion free, then the conjecture asserts that for any $\mathbb{Z}\Gamma$-module M, the equality $\mathrm{pd}_{\mathbb{Z}\Gamma'} M = \mathrm{pd}_{\mathbb{Z}\Gamma} M$ holds true.

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- **(Serre's Theorem, 1969)** Let Γ be a torsion free group and Γ' be its subgroup of finite index. Then $\mathrm{cd}\,\Gamma = \mathrm{cd}\,\Gamma'$, where $\mathrm{cd}\,\Gamma := \mathrm{pd}_{\mathbb{Z}\Gamma}\,\mathbb{Z}$ with trivial action.

Over $\mathbb{Z}\Gamma$, Moore's conjecture can be stated as follows:

(**Aljadeff-Meir**.) Let Γ be any group and Γ' a normal subgroup of finite index. Assume no elements of prime order lie in $(\Gamma - \Gamma')$. Then any $\mathbb{Z}\Gamma$-module M which is projective over $\mathbb{Z}\Gamma'$ will be also a projective $\mathbb{Z}\Gamma$-module.

Moore's condition is necessary in the conjecture in the following sense:

(**Aljadeff-Meir**.) Let Γ be a group and Γ' be its normal subgroup of finite index. If there are elements of prime order in $\Gamma - \Gamma'$, then there exists a $\mathbb{Z}\Gamma$-module M, projective over $\mathbb{Z}\Gamma'$ but not projective over $\mathbb{Z}\Gamma$.

$\mathbf{H}\mathfrak{F}$

We say that $\Gamma \in \mathbf{H}\mathfrak{F}$, if there is an ordinal number α such that $\Gamma \in \mathbf{H}_\alpha\mathfrak{F}$. The algebraic consequence of this is that there is a finite resolution of the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$;

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

in which $C_j = \oplus_i(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma_{j_i}} \mathbb{Z})$ such that $\Gamma_{j_i} \in \mathbf{H}_\beta\mathfrak{F}$, for some $\beta < \alpha$. The class $\mathbf{H}\mathfrak{F}$ has been defined by Kropholler.

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- We point out that the class $\mathbf{LH}\mathfrak{F}$ is a large class and it contains all abelian groups, all soluble by finite groups and also it is closed under subgroups and extensions.

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- (**Aljadeff, 2006**) Moore's conjecture holds for all groups in $\mathbf{LH}\mathfrak{F}$ under the condition that the module M is finitely generated.

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- (**Benson-Goodearl, 2000**) Let Γ be a group and Γ' be a subgroup of Γ of finite index. Let M be a flat $R\Gamma$-module which is projective over $R\Gamma'$. Then M is projective over $R\Gamma'$.

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- (**Aljadeff-Meir, 2012**) Moore's conjecture holds for any group in $\mathbf{LH}\mathfrak{F}$.

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- (**Aljadeff-Meir, 2012**) Moore's conjecture holds for any group in $\mathbf{LH}\mathfrak{F}$.
- There is a known example group which is not in $\mathbf{LH}\mathfrak{F}$ but Moore's conjecture is true for this group.

Conjecture A

(**Salarian-Bahlekeh**) Let (Γ, Γ', R) satisfy Moore's condition. Assume that M is an arbitrary $R\Gamma$-module, whose restriction to $R\Gamma'$ is injective. Then M is injective over $R\Gamma$.

theorem

(**Salarian-Bahlekeh**) Let Γ be an $\text{LH}\mathfrak{F}$-group and let the triple (Γ, Γ', R) satisfy Moore's condition. Then Conjecture A is true for (Γ, Γ', R).

proposition

Conjecture A recovers Moore's conjecture, in the sense that If Conjecture A holds true for a group Γ, then so does Moore's conjecture.

examples of Gorenstein projective modules

- If Γ is a finite group, then every $\mathbb{Z}\Gamma$-module which is $\mathbb{Z}$-free is Gorenstein projective. In particular, $\mathbb{Z}$ with the trivial Γ-action, is a Gorenstein projective $\mathbb{Z}\Gamma$-module.

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- Let Γ' be a subgroup of Γ of finite index, then any $\mathbb{Z}\Gamma$-module A which is projective as $\mathbb{Z}\Gamma'$-module is a Gorenstein projective $\mathbb{Z}\Gamma$-module.

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- Let Γ' be a subgroup of Γ of finite index, then any $\mathbb{Z}\Gamma$-module A which is projective as $\mathbb{Z}\Gamma'$-module is a Gorenstein projective $\mathbb{Z}\Gamma$-module.
- If Γ' is a subgroup of Γ and A is a Gorenstein projective $\mathbb{Z}\Gamma'$-module, then the induced module $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} A$ is a Gorenstein projective $\mathbb{Z}\Gamma$-module.

theorem

(**Salarian-Bahlekeh**) Let Γ be a group, Γ' be its subgroup of finite index. Assume that M is a $\mathbb{Z}\Gamma$-module. Then

- M is Gorenstein projective if and only if it is Gorenstein projective as a $\mathbb{Z}\Gamma'$-module.

theorem

(**Salarian-Bahlekeh**) Let Γ be a group, Γ' be its subgroup of finite index. Assume that M is a $\mathbb{Z}\Gamma$-module. Then

- M is Gorenstein projective if and only if it is Gorenstein projective as a $\mathbb{Z}\Gamma'$-module.
- M is Gorenstein injective if and only if it is Gorenstein injective as a $\mathbb{Z}\Gamma'$-module.

corollary

Let Γ be a group and Γ' be its subgroup of finite index. Let M be an $\mathbb{Z}\Gamma$-module. Then

- $\operatorname{Gpd}_{\mathbb{Z}\Gamma} M = \operatorname{Gpd}_{\mathbb{Z}\Gamma'} M$.

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Let Γ be a group and Γ' be its subgroup of finite index. Let M be an $\mathbb{Z}\Gamma$-module. Then

- $\operatorname{Gpd}_{\mathbb{Z}\Gamma} M = \operatorname{Gpd}_{\mathbb{Z}\Gamma'} M$.
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remark

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- (**Asadollahi-Bahlekeh-Salarian**). Let Γ be a group and Γ' be its subgroup of finite index. Then $\operatorname{Gcd}\Gamma = \operatorname{Gcd}\Gamma'$, where $\operatorname{Gcd}\Gamma = \operatorname{Gpd}_{\Gamma}\mathbb{Z}$ with trivial action.

Strongly Gorenstein flat modules

- A $\mathbb{Z}\Gamma$-module M is called *strongly Gorenstein flat*, if there exists an acyclic complex of projective $\mathbb{Z}\Gamma$-modules

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots,$$

with $M = \mathrm{Im}(P_0 \longrightarrow P_{-1})$ such that the functor $\mathrm{Hom}_{\mathbb{Z}\Gamma}(-, F)$ leaves the sequence exact, where F is a flat $\mathbb{Z}\Gamma$-module.

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- It follows from the definition that every strongly Gorenstein flat module is Gorenstein projective.

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- There is no known example of Gorenstein projective module which is not strongly Gorenstein flat.
- A finitely presented module M is Gorenstein projective if and only if it is strongly Gorenstein flat.

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- The trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ is strongly Gorenstein flat if and only if Γ is a finite group.
- The trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ is Gorenstein flat if and only if Γ is a finite group.
- For any $\mathbb{Z}\Gamma$-module M, one has $\mathrm{SGfd}_{\mathbb{Z}\Gamma} M \leq \mathrm{SGfd}_{\mathbb{Z}\Gamma} \mathbb{Z} + 1$.

proposition

Let Γ be a finite group and M be a $\mathbb{Z}\Gamma$-module. Then M is Gorenstein projective if and only if it is Gorenstein flat if and only if it is a strongly Gorenstein flat module.

theorem

Let Γ be a group, Γ' be its subgroup of finite index. Assume that M is a $\mathbb{Z}\Gamma$-module. Then

- M is Gorenstein flat if and only if it is Gorenstein projective as a $\mathbb{Z}\Gamma'$-module.

theorem

Let Γ be a group, Γ' be its subgroup of finite index. Assume that M is a $\mathbb{Z}\Gamma$-module. Then

- M is Gorenstein flat if and only if it is Gorenstein projective as a $\mathbb{Z}\Gamma'$-module.
- M is strongly Gorenstein flat if and only if it is strongly Gorenstein flat as a $\mathbb{Z}\Gamma'$-module.

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2. $\operatorname{Gcd} \Gamma < \infty$.
3. $\operatorname{silf} \Gamma < \infty$, whereas $\operatorname{silf} \Gamma$ is the supremum of injective lengths of flat $\mathbb{Z}\Gamma$-modules.

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3. $\operatorname{silf} \Gamma < \infty$, whereas $\operatorname{silf} \Gamma$ is the supremum of injective lengths of flat $\mathbb{Z}\Gamma$-modules.

Then the classes of Gorenstein projective $\mathbb{Z}\Gamma$-modules and strongly Gorenstein flat $\mathbb{Z}\Gamma$-modules are the same.

Thank you all