Ulrich ideals and modules

Shiro Goto

(Meiji University, Japan)

a joint work [GOTWY1]
with K. Ozeki, R. Takahashi,
K.-i. Watanabe, and K.-i. Yoshida

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1. Introduction

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Why the theory of MGMCM modules works so well?

Outline

2. Definitions

Let

- $(A, \mathfrak{m})$ a Cohen-Macaulay local ring, $d = \dim A \geq 0$
- I an $\mathfrak{m}$–primary ideal in A
- $I \supseteq Q$ as a reduction, where Q is a parameter ideal of A.

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(1)+(2) $\Longleftrightarrow$ $\mathrm{gr}_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a Cohen-Macaulay ring with $\mathrm{a}(\mathrm{gr}_I(A)) = 1 - d$

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If $I = \mathfrak{m}$, then (3) is automatically satisfied and

(1)+(2) $\Longleftrightarrow$ $A \neq$ a RLR, $\mathrm{v}(A) = \mathrm{e}(A) + \dim A - 1$.

2. Definitions (Cont.)

If I is an Ulrich ideal, then

- I/Q is also a free A/I-module, since $I/I^2 \cong Q/I^2 \bigoplus I/Q$.
- $$I/Q \cong (A/I)^{\oplus(n-d)}$$

  where $n = \mu_A(I)$ $(> d)$.

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- $$I/Q \cong (A/I)^{\oplus(n-d)}$$

  where $n = \mu_A(I)$ $(> d)$.

Therefore, since $I/Q \subseteq A/Q$, we get

$$\boxed{0 < n - d \le (n-d) \cdot \mathrm{r}(A/I) = \mathrm{r}_A(I/Q) \le \mathrm{r}(A/Q) = \mathrm{r}(A)}$$

where $\mathrm{r}_A(*)$ denotes the Cohen–Macaulay type.

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where $\mathrm{r}_A(*)$ denotes the Cohen–Macaulay type.
Hence, if A is a **Gorenstein** local ring, A/I is a **Gorenstein** ring and $Q : I = I$, so that

Ulrich ideals are **good** ideals in the sense of [GIW].

2. Definitions (Cont.)

Let M be a finitely generated A-module.

Definition 2.2. *M is an* **Ulrich A-module with respect to I**, *if*

(1) *M is a* **maximal** *Cohen-Macaulay A-module,*

(2) $\mathrm{e}_I^0(M) = \ell_A(M/IM)$, *and*

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Suppose that M is a **maximal** Cohen–Macaulay A-module. Then

$$\mathrm{e}_I^0(M) = \mathrm{e}_Q^0(M) = \ell_A(M/QM) \geq \ell_A(M/IM) \geq \ell_A(M/\mathfrak{m}M) = \mu_A(M).$$

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Hence

- Condition (2) $\iff QM = IM$.
- If $I = \mathfrak{m}$, in general we have $\boxed{\mathrm{e}_\mathfrak{m}^0(M) \geq \mu_A(M)}$

and

$$\boxed{\mathrm{e}_\mathfrak{m}^0(M) = \mu_A(M) \iff M \text{ is a } \textbf{MGMCM} \text{ module.}}$$

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(3) I/I^2 is A/I-free.

Definition 2.2. *M is an* **Ulrich A-module with respect to I**, *if*

(1) M is a Cohen-Macaulay A-module of dimension d,

(2) $IM = QM$, and

(3) M/IM is A/I-free.

Example

Let R be a Cohen-Macaulay local ring of dimension $d > 0$. Let $F = R^n$ for $n > 0$ and look at the **idealization** of F over R:

$$A = R \ltimes F.$$

Let $\mathfrak{q}$ be an **arbitrary** parameter ideal of R and put

$$I = \mathfrak{q} \times F, \quad Q = \mathfrak{q}A.$$

Then A is a d-dimensional Cohen-Macaulay local ring and I is an **Ulrich ideal** of A with Q a reduction.

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In the local ring

$$A = k[[X, Y, Z]]/(Z^2 - XY),$$

$\mathfrak{m} = (x, y, z)$ is a **unique** Ulrich ideal and $\mathfrak{p} = (z, x)$ is an Ulrich A–module with respect to $\mathfrak{m}$.

Theorem 3.2

Let I be an Ulrich ideal in A. Then for $\forall i \geq d$,

$$\mathrm{Syz}_A^i(A/I)$$

is an Ulrich A–module with respect to I.

Here $\mathrm{Syz}_A^i(A/I)$ denotes the i–th syzygy module of A/I in a minimal free resolution.

Proof of Theorem 3.2

- Assume that I is an $\mathfrak{m}$**–primary ideal** of A with Q a **minimal reduction**.
- Suppose $d > 0$, choose $a \in Q \setminus \mathfrak{m}Q$, and put

$$\overline{A} = A/(a), \ \ \overline{I} = I/(a), \ \ \overline{Q} = Q/(a).$$

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Fact 1

If I is an Ulrich ideal of A, then $\overline{I}$ is an Ulrich ideal of $\overline{A}$.

Proof.

The exact sequence

$$0 \to [I^2 + (a)]/I^2 \to I/I^2 \to \overline{I}/\overline{I}^2 \to 0$$

is split, since **I/I^2 is A/I-free**. □

Proof of Theorem 3.2, Cont.

Fact 2. [Vasconcelos [V]]

Suppose that I/I^2 is A/I-free. Then for $\forall i \geq 1$,

$$\mathrm{Syz}_A^i(A/I)/a{\cdot}\mathrm{Syz}_A^i(A/I) \cong \mathrm{Syz}_{\overline{A}}^{i-1}(\overline{A}/\overline{I}) \bigoplus \mathrm{Syz}_{\overline{A}}^i(\overline{A}/\overline{I}).$$

Proof of Theorem 3.2, Cont.

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$I/aI \qquad\qquad \overline{A}/\overline{I} \bigoplus \overline{I}$

Proof of Fact 2

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Let $I = (a) + (a_2, a_3, \ldots, a_n)$ with $n = \mu_A(I)$. Then

$$I/aI = A\overline{a} + \sum_{i=2}^{n} A\overline{a_i}.$$

Assume that $c\overline{a} + \sum_{i=2}^{n} c_i\overline{a_i} = 0$ in $\underline{I/aI}$ with $c, c_i \in A$. Then

$$c\overline{a} + \sum_{i=2}^{n} c_i\overline{a_i} = 0 \text{ in } \underline{I/I^2}.$$

Since $\{\overline{a}, \overline{a_i} \in I/I^2 \ (2 \le i \le n)\}$ forms **a free A/I-basis** of I/I^2, we get $c \in I$. Thus

$$c\overline{a} = \sum_{i=2}^{n} c_i\overline{a_i} = 0 \text{ in } I/aI,$$

so that $A\overline{a} \cong \overline{A}/\overline{I}$ and $I/aI \cong \overline{A}/\overline{I} \bigoplus \overline{I}$.

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is an Ulrich A–module with respect to I.

Proof of Theorem 3.2

$d = 0$:

- Look at the exact sequence $0 \to I \to A \to A/I \to 0$. Then $I^2 = (0)$ and $I \cong (A/I)^{\oplus n}$ $(n = \mu_A(I) > 0)$.
- Hence $\boxed{\mathrm{Syz}_A^i(A/I) \cong (A/I)^{\oplus n^i}}$ for $\forall i \geq 0$.

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$d > 0$:

Assume that our assertion holds true for $d - 1$. Let $i \geq d$. Then

- $\boxed{\mathrm{Syz}_A^i(A/I)/a{\cdot}\mathrm{Syz}_A^i(A/I) \cong \mathrm{Syz}_{\overline{A}}^{i-1}(\overline{A}/\overline{I}) \bigoplus \mathrm{Syz}_{\overline{A}}^i(\overline{A}/\overline{I})}$ and $\mathrm{Syz}_{\overline{A}}^{i-1}(\overline{A}/\overline{I})$, $\mathrm{Syz}_{\overline{A}}^i(\overline{A}/\overline{I})$ are Ulrich $\overline{A}$–modules w.r.t. $\overline{I}$.

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- Hence $\mathrm{Syz}_A^i(A)/a{\cdot}\mathrm{Syz}_A^i(A/I)$ is an Ulrich $\overline{A}$–module w.r.t. $\overline{I}$, so that $\boxed{\mathrm{Syz}_A^i(A/I) \text{ is an Ulrich } A\text{–module w.r.t. } I}$.

4. Relation between Ulrich ideals and modules

The converse of Theorem 3.2 is also true:

Theorem 4.1(cf. [BHU]). TFAE.

(1) *I is an Ulrich ideal of A.*

(2) $\mathrm{Syz}_A^i(A/I)$ *is an Ulrich A–module w.r.t. I for $\forall i \geq d$.*

(3) *There exists an exact sequence*

$$0 \to X \to F \to Y \to 0$$

of finitely generated A–modules such that

(a) *F is free,*

(b) *$X \subseteq \mathfrak{m}F$, and*

(c) *X and Y are Ulrich A–modules w.r.t. I.*

When $d > 0$, we can add the following.

(4) *$\mu_A(I) > d$, I/I^2 is A/I–free, and $\mathrm{Syz}_A^i(A/I)$ is an Ulrich A–module w.r.t. I for some $i \geq d$.*

The implication (3) $\Rightarrow$ (1) in Theorem 4.1 is based on the following.

Lemma 4.2

Let

$$0 \to X \to F \to Y \to 0$$

be an exact sequence of finitely generated A-modules and assume that

(a) *F is a free A-module,*

(b) *$X \subseteq \mathfrak{m}F$, and*

(c) *Y is an Ulrich A-module w.r.t. I.*

Then X is an Ulrich A-module w.r.t. $I \Leftrightarrow I$ is an Ulrich ideal of A.

Proof of (4) $\Rightarrow$ (1) in Theorem 4.1

Let $a \in Q \setminus \mathfrak{m}Q$ and put

$$\overline{A} = A/(a), \quad \overline{I} = I/(a), \quad \overline{Q} = Q/(a).$$

Proof of (4) $\Rightarrow$ (1) in Theorem 4.1

Let $a \in Q \setminus \mathfrak{m}Q$ and put

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Then

- $\boxed{\mathrm{Syz}_A^i(A/I)/a{\cdot}\mathrm{Syz}_A^i(A/I) \cong \mathrm{Syz}_{\overline{A}}^{i-1}(\overline{A}/\overline{I}) \bigoplus \mathrm{Syz}_{\overline{A}}^{i}(\overline{A}/\overline{I}).}$

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- Hence $\mathrm{Syz}_{\overline{A}}^{i-1}(\overline{A}/\overline{I})$ and $\mathrm{Syz}_{\overline{A}}^i(\overline{A}/\overline{I})$ are Ulrich modules w.r.t $\overline{I}$, so that $\overline{I}$ is an Ulrich ideal of $\overline{A}$ by [(3) $\Rightarrow$ (1)].
- Hence $I^2 \subseteq Q$, which yields $I^2 = QI$, because I/I^2 is A/I-free.

Remark 4.3 ([BHU])

Let $A = k[[X]]/(X^3)$. We look at the exact sequence

$$0 \to \mathfrak{m}^2 \to A \xrightarrow{x} A \to A/\mathfrak{m} \to 0,$$

where $\mathfrak{m} = (x)$. Then, since $\mathfrak{m}^3 = (0)$,

$$\mathfrak{m}^2 = \mathrm{Syz}_A^2(A/\mathfrak{m})$$

is an Ulrich module w.r.t $\mathfrak{m}$, but $\mathfrak{m}$ is not an Ulrich ideal of A, because $\mathfrak{m}^2 \neq (0)$.

This example shows that (4) $\Rightarrow$ (1) in Theorem 4.1 is not true in general, when $d = 0$.

Question 4.4

How many Ulrich ideals are contained in a given Cohen–Macaulay local ring?

How many Ulrich ideals?

Example

Look at the numerical semigroup ring

$$A = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]] \subseteq k[[t]] = \overline{A},$$

where $0 < a_1, a_2, \dots, a_\ell \in \mathbb{Z}$ such that $\mathrm{GCD}(a_1, a_2, \dots, a_\ell) = 1$.
Let $\mathcal{X}_A^g = \{$ Ulrich ideals I in A s.t. $I =$ (monomials in t)$\}$.
Then $\mathcal{X}_A^g$ **is finite**, and for example, we have the following.

(1) $\mathcal{X}^g_{k[[t^3,t^4,t^5]]} = \{\mathfrak{m}\}$.

(2) $\mathcal{X}^g_{k[[t^4,t^5,t^6]]} = \{(t^4, t^6)\}$.

(3) $\mathcal{X}^g_{k[[t^a,t^{a+1},\cdots,t^{2a-2}]]} = \emptyset$, if $a \geq 5$.

(4) Let $1 < a < b \in \mathbb{Z}$ such that $\mathrm{GCD}(a, b) = 1$. Then $\mathcal{X}^g_{k[[t^a,t^b]]} \neq \emptyset$ if and only if a or b is even.

(5) Let $A = k[[t^4, t^6, t^{4\ell-1}]]$ $(\ell \geq 2)$. Then $\sharp\mathcal{X}_A^g = 2\ell - 2$.

Proof of the finiteness of $\mathcal{X}_A^g$.

Remember that

$$I/Q \cong (A/I)^{\oplus(n-d)} \quad \text{and} \quad \frac{I}{a} = A[\frac{I}{a}]$$

where $Q = (a)$ and we have

$$A/I \subseteq I/Q \cong \frac{I}{a}/A = A[\frac{I}{a}]/A \subseteq \overline{A}/A.$$

Hence

$$A : \overline{A} \subseteq I$$

for every Ulrich ideal I of A. □

5. Minimal free resolutions of Ulrich ideals

Assume that I is an **Ulrich ideal** of A which contains a parameter ideal Q of A as a reduction. Let

$$\mathbb{F}_\bullet : \quad \cdots \to F_i \stackrel{\partial_i}{\to} F_{i-1} \to \cdots \to F_1 \stackrel{\partial_1}{\to} F_0 \to A/I \to 0$$

be a **minimal** free resolution of A/I and $\beta_i = \mathrm{rank}_A F_i \quad (i \geq 0)$ the i-th Betti number of A/I. We put $n = \mu_A(I)$.

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Theorem 5.1. *The following assertions hold true.*

(1) $A/I \otimes_A \partial_i = 0$ for $\forall i \geq 1$.

(2)

$$\beta_i = \begin{cases} (n-d)^{i-d} \cdot (n-d+1)^d & (i \geq d), \\ \binom{d}{i} + (n-d)\beta_{i-1} & (1 \leq i \leq d), \\ 1 & (i = 0) \end{cases}$$

for $i \geq 0$. Hence $\beta_i = \binom{d}{i} + (n-d)\beta_{i-1}$ for $\forall i \geq 1$.

5. Minimal free resolutions of Ulrich ideals (Cont.)

By the exact sequence

$$0 \to Q \to I \to (A/I)^{\oplus(n-d)} \to 0,$$

we get a free resolution of I which is the "direct sum" of minimal free resolutions of Q and A/I. This resolution is actually **minimal**.

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Corollary 5.2

A minimal free resolution of I is obtained by the "direct sum" of those of Q and $(A/I)^{\oplus(n-d)}$.

Corollary 5.3

$\mathrm{Syz}_A^{i+1}(A/I) \cong [\mathrm{Syz}_A^i(A/I)]^{\oplus(n-d)}$ *for all $i \geq d$. Hence*

$$\mathrm{Syz}_A^{i+1}(A/I) \cong \mathrm{Syz}_A^i(A/I)$$

for all $i \geq d$, if A is a Gorenstein local ring.

Example 5.4

Suppose that A is a Gorenstein local ring of dimension 1 and I an Ulrich ideal of A. Then $\mu_A(I) = 2$. We write $I = (a, x)$ ($x \in A$) with $Q = (a)$ is a reduction of I. Then $x^2 = ay$ for some $y \in I$, since $I^2 = aI$, and a minimal free resolution of A/I is given by

$$\mathbb{F}_\bullet : \cdots \to A^2 \xrightarrow{\begin{pmatrix} -x & -y \\ a & x \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} -x & -y \\ a & x \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} a & x \end{pmatrix}} A \xrightarrow{\varepsilon} A/I \to 0.$$

Example 5.5

Suppose that A is a Gorenstein local ring of dimension 2 and I an Ulrich ideal of A. Then $\mu_A(I) = 3$. Let $I = (a, b, x)$ ($x \in A$) with $Q = (a, b)$ a reduction of I. Hence $x^2 = ay + bz$ for some $y, z \in I$, and a minimal free resolution of I is given by

$$\cdots \to A^4 \xrightarrow{\begin{pmatrix} x & -z & y & 0 \\ b & -x & 0 & -y \\ -a & 0 & -x & -z \\ 0 & a & b & x \end{pmatrix}} A^4 \xrightarrow{\begin{pmatrix} b & -x & 0 & -y \\ -a & 0 & -x & -z \\ 0 & a & b & x \end{pmatrix}} A^3 \xrightarrow{\begin{pmatrix} a & b & x \end{pmatrix}} I \to 0.$$

5. Minimal free resolutions of Ulrich ideals (Cont.)

Let $\mathrm{I}_1(\partial_i)$ $(i \geq 1)$ be the ideal of A generated by the entries of the matrix ∂_i.

Theorem 5.6

$\mathrm{I}_1(\partial_i) = I$ *for* $\forall i \geq 1$.

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Theorem 5.6

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Proof.

- $\mathrm{I}_1(\partial_i) + Q = I$ for $\forall i \geq 1$ by induction on d.
- $\mathrm{I}_1(\partial_i) \supseteq Q$ for $1 \leq i \leq d$ by Corollary 5.2.
- $\mathrm{I}_1(\partial_{i+1}) = \mathrm{I}_1(\partial_i)$ for $\forall i \geq d+1$, since

$$[F_{i+1} \xrightarrow{\partial_{i+1}} F_i] = [F_i \xrightarrow{\partial_i} F_{i-1}]^{\oplus(n-d)}.$$

- $\mathrm{I}_1(\partial_{d+1}) \supseteq \mathrm{I}_1(\partial_d) = I$, since $\partial_{d+1} = \begin{pmatrix} * \\ \partial_d^{\oplus(n-d)} \end{pmatrix}$.

Hence $\mathrm{I}_1(\partial_i) = I$ for $\forall i \geq 1$. □

Corollary 5.7

Let I and J be Ulrich ideals of A. Then $I = J$ if and only if

$$\mathrm{Syz}_A^i(A/I) \cong \mathrm{Syz}_A^i(A/J)$$

for **some** $i \geq 0$.

Is $\mathcal{X}_A$ a finite set?

Let $\mathcal{X}_A = \{I \mid I \text{ is an } \textbf{Ulrich} \text{ ideal of } A\}$.

Theorem 5.8

Suppose that A is of finite C-M representation type. Then $\mathcal{X}_A$ is a finite set.

Proof of Theorem 5.8

- Let $\mathcal{Y}_A = \{[\mathrm{Syz}_A^d(A/I)] \mid I \in \mathcal{X}_A\}$, where $[\mathrm{Syz}_A^d(A/I)]$ denotes the **isomorphism class** of $\mathrm{Syz}_A^d(A/I)$.
- Let $I \in \mathcal{X}_A$ and $n = \mu_A(I)$.

Proof of Theorem 5.8

- Let $\mathcal{Y}_A = \{[\mathrm{Syz}_A^d(A/I)] \mid I \in \mathcal{X}_A\}$, where $[\mathrm{Syz}_A^d(A/I)]$ denotes the **isomorphism class** of $\mathrm{Syz}_A^d(A/I)$.
- Let $I \in \mathcal{X}_A$ and $n = \mu_A(I)$.

Then

$$\boxed{n - d \leq (n-d){\cdot}\mathrm{r}(A/I) = \mathrm{r}_A(I/Q) \leq \mathrm{r}(A)}$$

because $I/Q \cong (A/I)^{\oplus(n-d)}$.

Proof of Theorem 5.8

- Let $\mathcal{Y}_A = \{[\mathrm{Syz}_A^d(A/I)] \mid I \in \mathcal{X}_A\}$, where $[\mathrm{Syz}_A^d(A/I)]$ denotes the **isomorphism class** of $\mathrm{Syz}_A^d(A/I)$.
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Then

$$\boxed{n - d \leq (n-d)\cdot \mathrm{r}(A/I) = \mathrm{r}_A(I/Q) \leq \mathrm{r}(A)}$$

because $I/Q \cong (A/I)^{\oplus(n-d)}$. Hence

$$\boxed{\mu_A(\mathrm{Syz}_A^d(A/I)) = \beta_d^A(A/I) = (n-d+1)^d \leq (\mathrm{r}(A)+1)^d \ll \infty.}$$

Proof of Theorem 5.8

- Let $\mathcal{Y}_A = \{[\mathrm{Syz}_A^d(A/I)] \mid I \in \mathcal{X}_A\}$, where $[\mathrm{Syz}_A^d(A/I)]$ denotes the **isomorphism class** of $\mathrm{Syz}_A^d(A/I)$.
- Let $I \in \mathcal{X}_A$ and $n = \mu_A(I)$.

Then

$$\boxed{n - d \leq (n-d){\cdot}\mathrm{r}(A/I) = \mathrm{r}_A(I/Q) \leq \mathrm{r}(A)}$$

because $I/Q \cong (A/I)^{\oplus(n-d)}$. Hence

$$\boxed{\mu_A(\mathrm{Syz}_A^d(A/I)) = \beta_d^A(A/I) = (n-d+1)^d \leq (\mathrm{r}(A)+1)^d \ll \infty.}$$

Since A is of **finite C-M representation type**, $\mathcal{Y}_A$ is **finite**. Hence $\mathcal{X}_A$ is also **finite**, because $\mathcal{X}_A = \mathcal{Y}_A$ by Corollary 5.7.

Is $\mathcal{X}_A$ a finite set?

Theorem 5.8

Suppose that A is of finite C-M representation type. Then $\mathcal{X}_A$ is a finite set.

Is $\mathcal{X}_A$ a finite set?

Example

Let $A = k[[X, Y, Z]]/(Z^2 - XY)$. Then $\mathcal{X}_A = \{\mathfrak{m}\}$.

Is $\mathcal{X}_A$ a finite set?

Example

Let $A = k[[X, Y, Z]]/(Z^2 - XY)$. Then $\mathcal{X}_A = \{\mathfrak{m}\}$.

Proof.

The **indecomposable** maximal Cohen-Macaulay A-modules are A and $\mathfrak{p} = (z, x)$. We get $\boxed{\mathfrak{m} \in \mathcal{X}_A}$ since $\mathfrak{m}^2 = (x, y)\mathfrak{m}$.
Let $I \in \mathcal{X}_A$. Then $\mu_A(I) = 3$. Put $X = \mathrm{Syz}_A^2(A/I)$.

Then

$$X = \mathrm{Syz}_A^2(A/I) \cong \mathfrak{p} \bigoplus \mathfrak{p} \cong \mathrm{Syz}_A^2(A/\mathfrak{m}),$$

because $\mu_A(X) = 4$ and $\mathrm{rank}_A X = 2$. Hence $I = \mathfrak{m}$ by Corollary 5.7. □

Example 5.10

Example (One-dimensional Cohen-Macaulay local rings of finite CM-representation type)

(1) $\mathcal{X}_{k[[t^3,t^4]]} = \{(t^4, t^6)\}$.

(2) $\mathcal{X}_{k[[t^3,t^5]]} = \emptyset$.

(3) $\mathcal{X}_{k[[X,Y]]/(Y(X^2-Y^{2\ell+1}))} = \{(x, y^{2\ell+1}), (x^2, y)\}$, where $\ell \geq 1$.

(4) $\mathcal{X}_{k[[X,Y]]/(Y(Y^2-X^3))} = \{(x^3, y)\}$.

(5) $\mathcal{X}_{k[[X,Y]]/(X^2-Y^{2\ell})} =$
$$\{(x^2, y), (x - y^\ell, y(x + y^\ell)), (x + y^\ell, y(x - y^\ell))\},$$
where $\ell \geq 1$ and $\operatorname{ch} k \neq 2$.

Example 5.11

Example (N. Taniguchi)

(1) $\mathcal{X}_{k[[t^3,t^5]]} = \emptyset$.

(2) $\mathcal{X}_{k[[t^3,t^7]]} = \{(t^6 - ct^7, t^{10}) \mid 0 \neq c \in k\}$.

(3) $\mathcal{X}_{k[[t^{2q+i} \mid 1 \leq i \leq 2q]]} = \emptyset$ for $\forall q \geq 2$.

(4) $\mathcal{X}_{k[[t^{2q+i} \mid 0 \leq i \leq 2q-2]]} = \emptyset$ for $\forall q \geq 3$.

(5) $\sharp(\mathcal{X}_{k[[X,Y]]/(Y^n)}) = \infty$ for $\forall n \geq 2$.

Thank you very much for your attention!

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