Gorenstein Homology, Relative Pure Homology and Virtually Gorenstein Rings

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Contents

Def. (Warfield)

Let $\mathcal{S}$ be a class of R-modules. An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of R-modules and R-homomorphisms is called $\mathcal{S}$- pure exact if for all $U \in \mathcal{S}$ the induced R-homomorphism $\mathrm{Hom}_R(U, B) \to \mathrm{Hom}_R(U, C)$ is surjective.

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An R-module P is called $\mathcal{S}$- pure projective (resp. $\mathcal{S}$- copure projective) if for any $\mathcal{S}$-pure exact sequence (resp. $\mathcal{S}$-copure exact sequence) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced R-homomorphism $\mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C)$ is surjective.

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Also, an R-module F is called $\mathcal{S}$- pure flat (resp. $\mathcal{S}$- copure flat) if for any $\mathcal{S}$-pure exact sequence (resp. $\mathcal{S}$-copure exact sequence) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced R-homomorphism $F \otimes_R A \to F \otimes_R B$ is injective.

Def.

Let $\mathcal{F}$ (resp. $\mathcal{G}$) be a class of R-modules and M an R-module. An R-homomorphism $\phi : F \to M$ (resp. $\phi : M \to G$) where $F \in \mathcal{F}$ (resp. $G \in \mathcal{G}$) is called an $\mathcal{F}$- precover (resp. a $\mathcal{G}$-preenvelope) of M if for any $F' \in \mathcal{F}$ (resp. $G' \in \mathcal{G}$), the induced R-homomorphism $\operatorname{Hom}_R(F', F) \to \operatorname{Hom}_R(F', M)$ (resp. $\operatorname{Hom}_R(G, G') \to \operatorname{Hom}_R(M, G')$) is surjective.

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If $\phi : F \to M$ (resp. $\phi : M \to G$) is an $\mathcal{F}$-precover (resp. a $\mathcal{G}$-preenvelope) of M and any R-homomorphism $f : F \to F$ (resp. $f : G \to G$) such that $\phi f = \phi$ (resp. $f\phi = \phi$) is an automorphism, then ϕ is called an $\mathcal{F}$-cover (resp. a $\mathcal{G}$-envelope) of M.

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The class $\mathcal{F}$ (resp. $\mathcal{G}$) is called (pre)covering (resp. (pre)enveloping) if every R-module admits an $\mathcal{F}$-(pre)cover (resp. a $\mathcal{G}$-(pre)envelope).

Lemma

Let $\mathcal{S}$ be a class of R-modules and $\mathbf{X} = \cdots \to X_{i+1} \stackrel{d_{i+1}}{\to} X_i \stackrel{d_i}{\to} X_{i-1} \to \cdots$ an exact complex of R-modules. For each $i \in \mathbb{Z}$, set $\mathbf{X_i} := 0 \to \operatorname{Im} d_{i+1} \hookrightarrow X_i \twoheadrightarrow \operatorname{Im} d_i \to 0$. Then

1. $\operatorname{Hom}_R(U, \mathbf{X})$ is exact for all $U \in \mathcal{S}$ if and only if $\mathbf{X_i}$ is $\mathcal{S}$-pure exact for all $i \in \mathbb{Z}$.
2. $\operatorname{Hom}_R(\mathbf{X}, V)$ is exact for all $V \in \mathcal{S}$ if and only if $\mathbf{X_i}$ is $\mathcal{S}$-copure exact for all $i \in \mathbb{Z}$.

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Def.

Let $\mathcal{S}$ be a class of R-modules. An exact complex $\mathbf{X}$ of R-modules is said to be $\mathcal{S}$-pure exact (resp. $\mathcal{S}$-copure exact) if it satisfies the equivalent conditions of part (1) (resp. (2)) of the above Lemma.

Def.

Let M be an R-module and $\mathcal{F}$ (resp. $\mathcal{G}$) a class of R-modules. A left $\mathcal{F}$(resp. right $\mathcal{G}$)-resolution of M is a Hom $_R(\mathcal{F},-)$ (resp. Hom $_R(-,\mathcal{G})$) exact (not necessarily exact) complex

$$\mathbf{F}_\bullet = \cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

(resp.

$$\mathbf{G}^\bullet = 0 \longrightarrow M \longrightarrow G^0 \longrightarrow \cdots \longrightarrow G^n \longrightarrow G^{n+1} \longrightarrow \cdots)$$

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Let $\mathcal{F}$ be a precovering (resp. $\mathcal{G}$ be a preenveloping) class of R-modules. Then every R-module has a left $\mathcal{F}$(resp. right $\mathcal{G}$)-resolution.

Def.

Let $\mathcal{F}$ and $\mathcal{G}$ be two classes of R-modules. The functor $\text{Hom}_R(-,\sim)$ is called right balanced by $\mathcal{F} \times \mathcal{G}$, if for each R-module M there exists a $\text{Hom}_R(-,\mathcal{G})$ exact complex

$$\mathbf{F}_\bullet = \cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with $F_n \in \mathcal{F}$ for all $n \geqslant 0$ and a $\text{Hom}_R(\mathcal{F},\sim)$ exact complex

$$\mathbf{G}^\bullet = 0 \longrightarrow M \longrightarrow G^0 \longrightarrow \cdots \longrightarrow G^n \longrightarrow G^{n+1} \longrightarrow \cdots$$

with $G^n \in \mathcal{G}$ for all $n \geqslant 0$.

Let $\mathcal{F}$ be a precovering class and $\mathcal{G}$ a preenveloping class of R-modules. Assume that

$$\mathbf{F}_\bullet = \cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a left $\mathcal{F}$-resolution of an R-module M and

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is a right $\mathcal{G}$-resolution of an R module N. Set:

$$\mathbf{F}_\circ := \cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

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Let $\mathcal{F}$ be a precovering class and $\mathcal{G}$ a preenveloping class of R-modules. Assume that

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$$\mathbf{G}^{\circ} := 0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow G^n \longrightarrow \cdots.$$

Define $\operatorname{Ext}^i_{\mathcal{F}}(M,N) := H^i(\operatorname{Hom}_R(\mathbf{F}_{\circ},N))$ and $\operatorname{Ext}^i_{\mathcal{G}}(M,N) := H^i(\operatorname{Hom}_R(M,\mathbf{G}^{\circ}))$ for all $i \geqslant 0$.

Def.

We say a homology theory $\mathcal{T}$ is an $\mathcal{S}$-pure homology if an R-module M is projective (resp. injective; flat) in $\mathcal{T}$ if and only if it is $\mathcal{S}$-pure projective (resp. $\mathcal{S}$-pure injective; $\mathcal{S}$-pure flat).

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Question

Is Gorenstein homology an $\mathcal{S}$-pure homology for an appropriate class $\mathcal{S}$ of R-modules?

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In what follows, we denote the class of Gorenstein projective R-modules, the class of Gorenstein injective R-modules and the class of Gorenstein flat R-modules by $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$, respectively.

- Every Gorenstein projective R-module is $\mathcal{GP}$-pure projective.

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- Assume that R is a Noetherian ring. Then every $\mathcal{GP}$-pure projective R-module is Gorenstein flat.

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- Let R be a coherent ring. Then every $\mathcal{GP}$-pure flat R-module of finite Gorenstein flat dimension is Gorenstein flat.

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So, Our candidate for such class $\mathcal{S}$ is $\mathcal{GP}$

Def.

Let $\mathcal{F}$ (resp. $\mathcal{G}$) be a class of R-modules and $\mathcal{F}^{\perp}$ (resp. ${}^{\perp}\mathcal{G}$) denote the class of R-modules M with the property that $\mathrm{Ext}\,^1_R(F, M) = 0$ (resp. $\mathrm{Ext}\,^1_R(M, G) = 0$) for all R-modules $F \in \mathcal{F}$ (resp. $G \in \mathcal{G}$).

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Assume that R is a Noetherian ring of finite dimension. Then ${}^{\perp}\mathcal{GI} \subseteq \mathcal{GP}^{\perp}$.

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Let R be a Noetherian ring of finite dimension. Then every R-module M admits a $\mathcal{GP}$-pure right $\mathcal{GI}$-resolution.

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This generalizes the notion of virtually Gorenstein Artin algebras which was introduced by Beligiannis and Reiten (2007).

Thm.

Let R be a Noetherian ring of finite dimension. The following are equivalent:

1. $\mathrm{Hom}_R(-,\sim)$ is right balanced by $\mathcal{GP} \times \mathcal{GI}$.
2. A short exact sequence $\mathbf{X} = 0 \to X_1 \to X_2 \to X_3 \to 0$ of R-modules is $\mathcal{GP}$-pure exact if and only if it is $\mathcal{GI}$-copure exact.
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3. R is virtually Gorenstein.

Cor.

Let R be a virtually Gorenstein ring. Then $\mathrm{Ext}^i_{\mathcal{GP}}(M,N) \cong \mathrm{Ext}^i_{\mathcal{GI}}(M,N)$ for all R-modules M and N and all $i \geq 0$.

Thm.

Let R be a Noetherian ring of finite dimension. The following are equivalent:

1. R is virtually Gorenstein.
2. The classes of Gorenstein injective and $\mathcal{GP}$-pure injective R-modules are the same.
3. Gorenstein homology is a $\mathcal{GP}$-pure homology.

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- Let $(R, \mathfrak{m})$ be any local ring with $\mathfrak{m}^2 = 0$. Then R is a virtually Gorenstein ring which is not Gorenstein. So, we have plenty examples of virtually Gorenstein rings which are not Gorenstein.
- Let k be a field. Then $R := k[x, y, z]/\langle x^2, yz, y^2 - xz, z^2 - yx\rangle$ is not virtually Gorenstein. Hence, $\mathrm{Hom}_R(-, \sim)$ is not right balanced by $\mathcal{GP} \times \mathcal{GI}$.

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Thank You