An investigation on Hankel Matrices

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Introduction

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d-dimensional Rational normal Scroll

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$m \times n$ Hankel Matrix

Consider the following Hankel Matrix:

$$\mathcal{H}_{m,n} := \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\ x_2 & x_3 & x_4 & \cdots & x_n & x_{n+1} \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ x_m & x_{m+1} & x_{m+2} & \cdots & x_{m+n-2} & x_{m+n-1} \end{pmatrix}$$

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Gruson-Peskine Trick

$I_t(H_{m,n}) = I_t(H_{\alpha,\beta})$ where $m + n = \alpha + \beta$

Objective

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Let $R := k[\mathbf{x}] = k[x_0, \cdots, x_n]$. The rational map

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Square Hankel Matrix

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Algebraic invariants: Codim, Multiplicity, Primary decomposition (its radical, unmixed part),Linear Syzygies,....

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Lemma

The following holds, for $0 \leq j \leq 2n-2$:

$$\frac{\partial f}{\partial x_j} = f_j = \begin{cases} \sum \alpha_{i,j}[1, \cdots, \widehat{(i+1)}, \cdots, (j\widehat{+2-}i), \cdots, n+1] & j < n, \\ \sum \beta_{i,j}[1, \cdots, (i+\widehat{1+j}-n), \cdots, (n\widehat{+2-}i), \cdots, n+1] & j \geq n. \end{cases}$$

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Lemma

Set $P := I_{n-1}(\mathcal{H})$. Then P is a prime ideal and the multiplicity of R/P is

$$e(R/P) = \frac{1}{3!}(n-1)n(n+1).$$

A fundamental primary component of J

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Lemma

Letting $J' = \{\mathrm{in}(f_0), \ldots, \mathrm{in}(f_{2n-2})\}$ *in the rev.lex. monomial order, one has an inequality* $e(R/J') < 2e(R/P)$.

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$$\ell(R_P/J_P) = \ell(R_P/P_P)$$

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so that P is the $P-$primary component of J.

Partial order on maximal minors

Usual ordering on maximal $n-1$-minors :

$[i_1, ..., i_{n-1}] \leq [j_1, ..., j_{n-1}] \Leftrightarrow i_1 \leq j_1, ..., i_{n-1} \leq j_{n-1}$.

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$e(R/J) = e(R/P) = 1/6(n-1)n(n+1)$, *and* $\quad \mathrm{ht}(J : P) \geq 4$

Linear rank

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The Linear Rank of an Ideal

Suppose that R is a standard graded ring and that I is generated by forms $\mathbf{f} = f_0, \cdots, f_n$ of a fixed degree s.

$$R(-(s+1))^{\ell} \oplus \sum_{j \geq 2} R(-(s+j)) \xrightarrow{\varphi} R(-s)^{n+1} \to I \to 0$$

The linear rank of $I = \ell$.

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The Eagon-Northcott Complex of P

This complex is better visualized as a symmetrization of the Koszul complex:

$$\bigwedge^{n+1} R^{n+1} \otimes_R Sym^2(R^{n-1})^* \to \bigwedge^{n} R^{n+1} \otimes_R Sym^1(R^{n-1})^* \xrightarrow{\text{EN}} \bigwedge^{n-1} R^{n+1} \otimes_R R \to I_{n-1}(H) \to 0$$

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$$(\gamma_0 x_0 \quad \gamma_1 x_1 \quad \cdots \quad \gamma_{2n-2} x_{2n-2})$$

The Hessian of the Hankel determinant

Hankel matrix has non-vanishing Hessian

$$\mathrm{Hess}(f) = \det \begin{pmatrix} & & \cdots & & f_{0,(2n-2)} \\ & & \cdots & f_{1,(2n-3)} & \\ & \cdots & f_{2,(2n-4)} & & \\ \vdots & \vdots & \ldots & & \vdots \\ f_{(2n-2),0} & \cdots & & & \end{pmatrix} \neq 0$$

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The polynomial f is called Homaloidal if its polar map is a Cremona map.

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$$\mathcal{H} = \mathcal{H}_3 = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$$

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Properties

(a) Let $P := I_2(\mathcal{H}) \subset R$; and $\mathfrak{m} := (x_0, \ldots, x_4)$, then $Ass(J) = \{P, \mathfrak{m}\}$ and $J^{unm} = P = J^{sat}$.

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(b) J is an ideal of of linear type and the syzygy matrix of J has linear rank 3; in particular, $\det \mathcal{H}$ is not homaloidal.

Modification of Criterion of Birationality of DHS, 2012

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Case of ideal of Linear type

If $I = (\nabla f)$ is of linear type then the following conditions are equivalent:

a. f is homaloidal.

b. I has maximal linear rank.

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Conjecture

$JP^i : P^{i+1} = I_{n-2-i}(\mathcal{H})$

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Corollary

It follows from birationality criterion that J is not Homaloidal.

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- $\mathrm{Hess}(f) \neq 0$
- J is generated by algebraically independent elements.

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Thanks for your attention