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The Aluffi Algebra of Projective Points

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Let X be a set of points in $\mathbb{P}^n_k$. The points of X are in general linear position: if $s \leq n$ then it means that the points span a $\mathbb{P}^{s-1}_k$, while if $s \geq n+1$ then it means that no subset of $n+1$ points of X is contained in a hyperplane of $\mathbb{P}^n_k$.

Proposition (Results for arbitrary n)

Let $J \subset R$ denote the ideal of $1 \leq s \leq n+1$ points in general linear position in $\mathbb{P}^n$. The pair $J \subset I = (J, I_n(\Theta))$ is torsion-free if and only if $s \neq 2$.

Example

Let $P_1 = [0:1:0]$, $P_2 = [0:0:1]$, $P_3 = [0:1:1] \in \mathbb{P}^2$. These are on the hyperplane $H = \{x_0 = 0\}$ and are not in general linear position. The ideal of these points is $J = (x_0, x_1^2 x_2 - x_1 x_2^2)$ which is not torsion-free.

Theorem (Explicit Generators)

Let X be a set of $n+2 \leq s \leq 2n$ points in general linear position in $\mathbb{P}_k^n$. Then the corresponding ideal J of points is minimally generated by $\binom{n+2}{2} - s$ quadrics of the form

$$g_{ij} = x_i x_j + \sum_{t=2n-s+1}^{n-1} \alpha_{ij}^{(t)} x_t x_n, \quad 0 \leq i < j \leq n,$$

such that $(i,j) \neq (t,n)$, for $2n-s+1 \leq t \leq n-1$. The coefficients $\alpha_{ij}^{(t)} \in k$ are uniquely determined by coordinates of points in X.

Example (Exceptions in $\mathbb{P}^2$)

Consider the following four points in general linear position in the projective plane: $(1:0:0),(0:1:0),(0:0:1),(1:1:1)$. *An easy calculation gives its defining ideal* $J=(xz-yz, xy-yz)$, *while the Jacobian matrix of these 2-forms is:*

$$\Theta = \begin{pmatrix} z & -z & x-y \\ y & x-z & -y \end{pmatrix}$$

Therefore,
$I := (J, I_2(\Theta)) = (xy-xz, xz-yz, xz+yz-z^2, -xy+y^2-yz, -x^2+xy+xz)$.
A computation with Macaulay *gives that* JI^2 *is minimally generated by* 11 *quartics, while* JI *is obviously generated by at most* 10 *quartics. Therefore, the pair* $J \subset I$ *is not torsion-free.*

This example has a curious algebro-geometric background.

Let ϕ denote the 5×5 skew-symmetric matrix whose Pfaffians are the generators of I. Pick a new set of indeterminates $\mathbf{T} = \{T_1, T_2, T_3, T_4, T_5\}$ (think of them as the homogeneous coordinates of $\mathbb{P}^4_k$) and consider the entries of the matrix product $\mathbf{T} \cdot \phi$. Next take the Jacobian matrix ψ of these bihomogeneous polynomials of bidegree $(1,1)$ with respect to x, y, z – the so-called *Jacobian dual matrix* of ϕ. Note that this a 5×3 matrix whose entries are linear forms in $k[\mathbf{T}]$.

Theorem

With the above notation we have:

$$\mathcal{R}_R(I) \simeq R[\mathbf{T}]/(I_1(\mathbf{T}.\phi), I_3(\psi)).$$

Thus, I is an ideal of fiber type. Moreover, $\mathcal{R}_R(I)$ has depth 1.

Corollary (Presentation of Aluffi Algebra)

With the above notation we have:

$$\mathcal{A}_{R/J}(I/J) \simeq R[\mathbf{T}]/(I_1(\mathbf{T}, \bar{\phi}), I_3(\bar{\psi})).$$

where $\bar{\phi}$ denotes the submatrix of ϕ consisting of the three last rows and $\bar{\psi}$ is obtained from ψ by evaluating both T_0, T_1 at zero

(When is the critical locus a power?)

Consider the 6 *points in* $\mathbb{P}_k^3$*, which are written as columns of the following matrix:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

We can see that $\mathfrak{m}^3 \neq I_3(\Theta)$*, but the pair* $J \subset I$ *is Aluffi torsion-free.*

This is a counterexample for the following conjecture:

Conjecture (Abbas-Rashid, 2011)

Let J *denote the ideal generated by quadrics in a polynomial ring* S*, such that* $r = \mathrm{ht}\,(J) \geq 2$*. Then, the following conditions are equivalent.*

(a) $I_r(\Theta) = \mathfrak{m}^r$.

(b) $I_r(\Theta)$ *is* $\mathfrak{m}$*-primary.*

Proposition (Points in $\mathbb{P}^n$ $(n \geq 3)$)

Let Γ be a set of $n+2$ distinct points in general position in $\mathbb{P}^n_k$ with $n \geq 3$. Let $J \subset R = k[x_0, \ldots, x_n]$ be the corresponding ideal of points Then $I_n(\Theta) = \mathfrak{m}^n$; in particular the pair $J \subset (J, I_n(\Theta))$ is torsion-free.

Proposition (Points in $\mathbb{P}^n$ $(n \geq 4)$)

Let Γ be a set of $n+3$ distinct points in general position in $\mathbb{P}^n_k$ with $n \geq 4$. Let $J \subset R = k[x_0, \ldots, x_n]$ be the defining ideal of Γ and let $I = (J, I_r(\Theta))$ stand for the Jacobian ideal of J. Then the pair $J \subset I$ is Aluffi torsion-free.