On Hibi rings

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Posets

Let P be a finite poset.

Let $x, y \in P$. It is said that x covers y, denoted $x \gtrdot y$, if $x > y$ and there exists no $z \in P$ such that $x > z > y$.

- A nonempty subposet C of P which is totally ordered is called a chain in P.
- The length of C is defined to be $|C| - 1$, and denoted $\ell(C)$.
- If every maximal chain of P has the same length, then P is called pure.
- The rank of P, denoted $\mathrm{rank P}$, is defined to be the maximal length of a chain in P.

Posets

- Let $x \in P$. Then $\mathrm{height_P(x)}$ (resp. $\mathrm{depth_P(x)}$) is defined to be the maximal length of a chain descending (resp. ascending) from x in P.
- Let $\hat{P}$ be the poset $P \cup \{\infty, -\infty\}$ with $-\infty < x < \infty$ for all $x \in P$.
- In the case that $P = \hat{Q}$ for some poset Q, we omit the lower index and simply write $\mathrm{height(x)}$ and $\mathrm{depth(x)}$.

Posets

Let $x, y \in P$.

- An upper bound of x, y is an element $z \in P$ such that $z \geq x$ and $z \geq y$.
- If the set $\{z \in P : z \text{ is an upper bound of } x \text{ and } y\}$ has a least element, this is obviously unique, is called the join of x and y, and it is denoted by $x \vee y$.
- By duality, one defines the meet $x \wedge y$ of two elements x, y in a poset.

Lattices

A lattice L is a poset with the property that for any $x, y \in L$, $x \vee y$ and $x \wedge y$ exist.

Distributive Lattices

Let L be a lattice. Then L is called distributive if satisfies one of the equivalent conditions:

- For any $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
- For any $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

A subset α of a poset P is called an order ideal or poset ideal if it satisfies the following condition:

- for any $x \in \alpha$ and $y \in P$, if $y \leq x$, then $y \in \alpha$.

The set of all order ideals of P is denoted by $\mathcal{I}(P)$. The union and intersection of two order ideals are obviously order ideals. Therefore, $\mathcal{I}(P)$ is a distributive lattice with the union and intersection.

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Distributive Lattices

An element $\alpha \in L$ is called join-irreducible if $\alpha \neq \min L$ and whenever $\alpha = \beta \vee \gamma$, then $\alpha = \beta$ or $\alpha = \gamma$.

Theorem (Birkhoff's famous theorem)

Let L be a finite distributive lattice, and P its subposet of join-irreducible elements. Then L is isomorphic to $\mathcal{I}(P)$.

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Hibi rings

Let K be a field. The Hibi ring of L over K is the K-algebra $K[L]$ generated by the elements $\alpha \in L$ and with the defining relations $\alpha\beta - (\alpha \wedge \beta)(\alpha \vee \beta)$.

Theorem

Let L be a finite distributive lattice. Then the generators of I_L form the reduced Gröbner basis of I_L with respect to a suitable order.

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Theorem

Let L be a finite distributive lattice. Then the generators of I_L form the reduced Gröbner basis of I_L with respect to a suitable order.

Theorem (Hibi - 1987)

Let L be a finite distributive lattice. Then $K[L]$ is isomorphic to the toric ring generated over K by the elements u_α with $\alpha \in L$, where u_α is the monomial $s\prod_{x\in\alpha} t_x$ in the polynomial ring $K[s, t_x : x \in P]$.

Theorem (Hibi - 1987)

Let L be a finite distributive lattice. Then $K[L]$ is a normal Cohen-Macaulay domain.

Hibi rings

A map $v : \hat{P} \to \mathbb{Z}_{\geq 0}$ is called order reversing if $v(x) \leq v(y)$ for all $x, y \in \hat{P}$ with $x \geq y$.

And v is a called strictly order reversing if $v(x) < v(y)$ for all $x, y \in \hat{P}$ with $x > y$.

We denote by $\mathcal{S}(\hat{P})$ the set of all order reversing functions v on $\hat{P}$ with $v(\infty) = 0$, and by $\mathcal{T}(\hat{P})$ the set of all strictly order reversing functions v on $\hat{P}$ with $v(\infty) = 0$.

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Hibi rings

Theorem (Hibi - 1987)

The toric ring $K[L]$ has a K-basis consisting of the monomials

$$s^{v(-\infty)} \prod_{x \in P} t_x^{v(x)}, \quad v \in \mathcal{S}(\hat{P}).$$

Theorem (Hibi - 1987)

The canonical module $\omega_L \subset K[L]$ has a K-basis consisting of the monomials

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Hibi rings

On $\mathcal{T}(\hat{P})$ one defines the following partial order. For $v, v' \in \mathcal{T}(\hat{P})$, we set $v \geq v'$ if the following conditions hold:

- for all $p \in \hat{P}$, $v(p) \geq v'(p)$, and
- the function $v - v' \in \mathcal{S}(\hat{P})$, where $v - v' : \hat{P} \to \mathbb{Z}_{\geq 0}$ is defined by $(v - v')(p) = v(p) - v'(p)$ for all $p \in \hat{P}$.

The minimal generators of ω_L are in one-to-one correspondence with the minimal elements of the poset $\mathcal{T}(\hat{P})$.

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Theorem (Hibi - 1987)

Let L be a finite distributive lattice. Then $K[L]$ is Gorenstein if and only if the poset P of join-irreducible elements of L is pure.

There have been some attempts to characterize the Hibi rings with the following properties in terms of the underlying distributive lattice and its poset of join-irreducible elements.

- Linear relations
- Linear resolution
- Pure resolution

Hibi rings

Let $R = S/I$, where I is a graded ideal of S such that R is Cohen-Macaulay. Then R is Gorenstein if and only if ω_R is a cyclic module, and hence generated in a single degree. The condition on ω_R may be weakened in different ways:

- If one only requires that the generators of ω_R are of the same degree, then R is called a level ring.
- If one requires that there is only one generator of least degree, then we call R a pseudo-Gorenstein ring.

Theorem (Ene - Herzog - Hibi - me - 2014)

The distributive lattice L is pseudo-Gorenstein if and only if

$$\mathrm{depth}(x) + \mathrm{height}(x) = \mathrm{rank}\hat{P} \quad \text{for all} \quad x \in P.$$

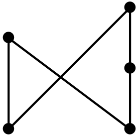

Figure: $\mathcal{I}(P)$ is not pseudo-Gorenstein

Pseudo-Gorenstein Hibi rings

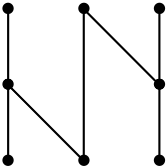

Figure: $\mathcal{I}(P)$ is pseudo-Gorenstein

Theorem (Miyazaki - 2007)

If for all $x \in P$ all maximal chains ascending from x have the same length, or for all $x \in P$ all maximal chains descending from x have the same length, then $K[L]$ is level.

We call a poset P with the above property a Miyazaki poset.

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Level Hibi rings

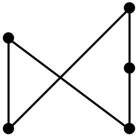

Figure: Level but not Miyazaki

Theorem (Ene - Herzog- Hibi - me - 2014)

Suppose $K[L]$ is level. Then

$$\mathrm{height}(x) + \mathrm{depth}(y) \leq \mathrm{rank}\hat{P} + 1$$

for all $x, y \in P$ with $x \gtrdot y$.

Hyper-planar lattices

Let L be a finite distributive lattice and P its poset of join-irreducible elements. We call L a hyper-planar lattice, if P as a set is the disjoint union of chains $C_1, \ldots, C_d$, where each C_i is a maximal chain in P. We call such a chain decomposition canonical.

An element $x \in C_i$ may be comparable with an element $y \in C_j$ for some $j \neq i$. If this is the case and if $x \gtrdot y$, then we call the chain $x \gtrdot y$ (of length one) a diagonal of P (with respect to the given canonical chain decomposition).

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Hyper-planar lattices

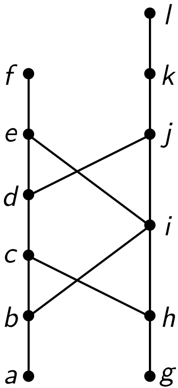
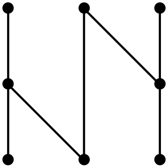


Figure: The poset of a planar lattice

Hyper-planar lattice



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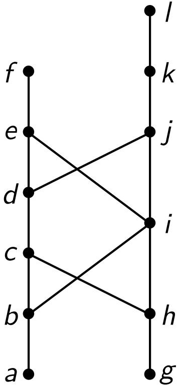

Figure: A canonical chain decomposition is not uniquely determined.

Regular hyper-planar lattices

We say that L is a regular hyper-planar lattice, if for any canonical chain decomposition $C_1 \cup C_2 \cup \ldots \cup C_d$ of P, and for all $x < y$ with $x \in C_i$ and $y \in C_j$ it follows that $\mathrm{height}_{C_i}(x) < \mathrm{height}_{C_j}(y)$.

Theorem (Ene - Herzog - Hibi - me - 2014)

Let L be a regular hyper-planar lattice and $C_1 \cup \ldots \cup C_d$ be a canonical chain decomposition of P. Then $K[L]$ is pseudo-Gorenstein if and only if all C_i have the same length.

Regular hyper-planar lattices

Theorem (Ene - Herzog - Hibi - me - 2014)

Let L be a regular hyper-planar lattice and let $C_1 \cup C_2 \cup \cdots \cup C_d$ be a canonical chain decomposition of P. We assume that all C_i have the same length. TFAE:

(a) L is Gorenstein;

(b) L is level;

(c) P is a Miyazaki poset.

Regular hyper-planar lattices

Theorem (Ene - Herzog - Hibi - me - 2014)

Let L be a regular planar lattice. TFAE:

(a) L is level;

(b) $\mathrm{height(x)} + \mathrm{depth(y)} \leq \mathrm{rank\hat{P}} + 1$ for all $x, y \in P$ with $x \gtrdot y$;

(c) for all $x, y \in P$ with $x \gtrdot y$, either $\mathrm{depth(y)} = \mathrm{depth(x)} + 1$ or $\mathrm{height(x)} = \mathrm{height(y)} + 1$.

Butterfly poset

Let P be a finite poset with a canonical chain decomposition $C_1 \cup C_2$ with $2 \leq |C_1| \leq |C_2|$. For $i = 1, 2$, let x_i be the maximal and y_i the minimal element of C_i. We call P a butterfly poset (of type (C_1, C_2)), if $x_1 \gtrdot y_2$ and $x_2 \gtrdot y_1$ are the only diagonals of P.

Butterfly poset

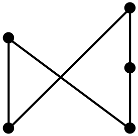

Figure: A Butterfly poset

Butterfly poset

Theorem (Ene - Herzog - Hibi - me - 2014)

Let P be a butterfly poset of type (C_1, C_2), and L its ideal lattice. Then the following conditions are equivalent:

(a) $T/\mathrm{in}_<(I_L)$ is level;

(b) L is level;

(c) $\mathrm{height}(x) + \mathrm{depth}(y) \leq \mathrm{rank}\hat{P} + 1$ for all $x, y \in P$ with $x \gtrdot y$;

(d) for all $x, y \in P$ with $x \gtrdot y$, either $\mathrm{depth}(y) = \mathrm{depth}(x) + 1$ or $\mathrm{height}(x) = \mathrm{height}(y) + 1$;

(e) $|C_1| = 2$.

One diagonal poset

Theorem (Ene - Herzog - Hibi - me - 2014)

Let L be a simple planar lattice whose poset P of join-irreducible elements has the single diagonal $x \gtrdot y$ with respect to a canonical chain decomposition. Then the following conditions are equivalent:

(a) L is level;

(b) $\mathrm{height}(x) + \mathrm{depth}(y) \leq \mathrm{rank}\hat{\mathrm{P}} + 1$;

(c) either $\mathrm{depth}(y) = \mathrm{depth}(x) + 1$ or $\mathrm{height}(x) = \mathrm{height}(y) + 1$.

One diagonal poset

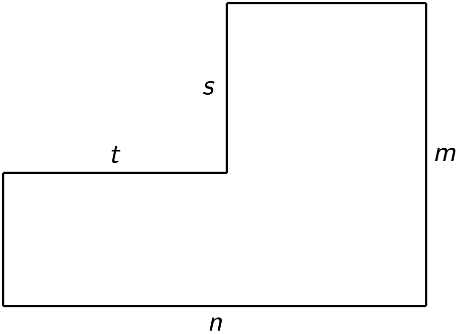

Figure: A ladder

One diagonal poset

Theorem (Ene - Herzog - Hibi - me - 2014)

Let I be the ladder determinantal ideal of 2-minors of a ladder. Then S/I is level if and only if $\min\{m, n\} \leq s + t$.

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Thanks for your attention.