Polynomial Wavelet Type Expansions on the Spline

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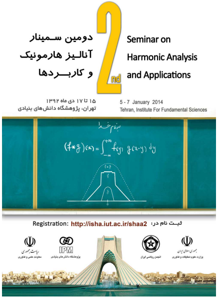

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Introduction

We present a polynomial wavelet-type system on S^d such that any continuous function can be expanded with respect to these wavelets.
The order of the growth of the degree of the polynomials is optimal.
The coefficients in the expansion are the inner product of the function and the corresponding element of a dual wavelet system.
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The dual wavelet system is also a polynomial system with the same growth of degree of polynomials. The system is redundant.

A construction of a polynomial basis is also presented. In constrast to our wavelet-type system, this basis is not suitable for implementation, because of two drawbacks: first there are no explicit formoula for the coefficient functionals and, secend, the growth of the degree of polynomials is too rapid.

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- π_n^d : is the space of polynomials in d variables of degree at most n.
- $B^d = \left\{ x \in R^d \| \|x\| \leq 1 \right\}, \quad S^d = \partial B^{d+1}.$

- $G_n^{(\lambda)}$: Denotes the n^{th} **Gegenbauer polynomial** of order λ. **Jacobi's polynomials**, $\{P_n^{(\alpha,\beta)}(x)\}$, are defined as orthogonal polynomials with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ on $(-1,1)$. $(\alpha > -1,\ \beta > -1)$
  If we put $\alpha = \beta$ in the above definition and define $A_0^{(\lambda)} = 1$,

$$A_n^{(0)} = \frac{2^{n+1}(n-1)}{1.3.\ldots.(2n-1)},\ n \geq 1$$

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  then **Gegenbauer polynomials** are defined as

$$G_n^{(\lambda)} = A_n^{(\lambda)} P_n^{(\alpha,\alpha)}(x),\ \lambda = (\alpha + \frac{1}{2}),\ n \geq 0.$$

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- $\bigoplus_{n=1}^{\infty} H_n^d$ is dense in $L_2(S^d)$.
- **Muller**'s formula (1966): Let $\{Y_{nk}, k = 1, \cdots \delta_n^d$ be an orthonormal basis for H_n^d, then

$$\sum_{k=0}^{\delta_n^d} Y_{nk}(x)Y_{nk}(y) = \frac{(2n+d-1)}{\omega_d(d-1)} G_n^{\frac{d-1}{2}}(x.y) \quad (1)$$

for all $x, y \in S^d$ and for all $n = 0, 1, \ldots$.

Theorem 1: F. Mahaskar and others (2001)

There is constants N_d and A_d depending only on d so that for any finite set $\{\eta_l\}_{l\in\Omega}$ of distinct points $\eta_l \in S^d$ and for any positive integer $N \geq N_d$ satisfying

$$N \max_{x\in S^d} \min_{l\in\Omega} |x - \eta_l| \leq A_d$$

there exist nonnegative weights $a_l, l \in \Omega$ such that

$$\int_{S^d} P(x) ds(x) = \sum_{l\in\Omega} a_l P(\eta_l)$$

for all $P \in \pi_N^{d+1}$.

Because of this theorem, to each positive integer j we can assign a set $\{\eta_l^{(j)}\}_{l\in\Omega_j}$ of distinct points $\eta_l^{(j)} \in S^d$ and a set $\{a_l^{(j)}\}_{l\in\Omega_j}$ of nonnegative weights the following properties:
card$(\Omega_j)\sim 2^{d_j}$,

$$\int_{S_d} P(x)ds(x) = \sum_{l\in\Omega_j} a_l^{(j)} P(\eta_l^{(j)})$$

for any $P \in \pi_{2^j}^{d+1}$. Additionally, we introduce the set $\Omega_0 := \{0\}$ and put $a_0^{(0)} = \omega_d$.

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- card$(\Omega_j)\sim 2^{d_j}$.
- $$\int_{S^d} P(x)ds(x) = \sum_{l\in\Omega_j} a_l^{(j)} P(\eta_l^{(j)}) \tag{2}$$

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- Let $h_j(n) = \frac{\binom{2^j-n+d-1}{d}}{\binom{2^j+d-1}{2^j-1}}$ for $n = 0, ..., 2^j - 1$, $h_j(n) = 0$ for $n = 2^j, 2^j + 1, ...$

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- An inequality due to kogbeliantz states that

$$\sum_{n=0}^{n} \binom{N-n+d}{d} (2n+d-1) G_n^{(\frac{d-1}{2})}(t) \geq 0, \; for\ all\ t \in [-1,1]. \quad (3)$$

Now we are going to define our **wavelet-type** functions.
Set for $j = 1, ..., n = 0, 1, ...$:

$$g_j(n) = h_j(n) + h_{j-1}(n)$$
$$\tilde{g}_j(n) = h_j(n) - h_{j-1}(n)$$

$g_0(0) = h_0(0) + 1$, $\tilde{g}_0(0) = h_0(0) - 1$, $g_0(n) = \tilde{g}_0(n) = 0$, for $n = 1, 2, ...$

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For each nonnegative integer j and for each $l \in \Omega_{j+1}$, define the wavelet function Ψ_{jl}, the dual wavelet function $\widetilde{\Psi}_{jl}$ and the scaling function $\Phi_{(j+1)l}$ by

$$\Psi_{jl}(x) = \frac{1}{\omega_d(d-1)} \sum_{n \in z^+} g_j(n)(2n+d-1) \times G_n^{(\frac{d-1}{2})}(\eta_l^{(j+1)}. x)$$

$$\widetilde{\Psi}_{jl}(x) = \frac{1}{\omega_d(d-1)} \sum_{n \in z^+} \tilde{g}_j(n)(2n+d-1) \times G_n^{(\frac{d-1}{2})}(\eta_l^{(j+1)}. x)$$

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Complete this collection by the function $\Phi_{0,0} = \frac{1}{\omega_d}$ and set $\Phi_0 = \sqrt{\omega_d}\Phi_{0,0}$. For $F \in C(S^d)$, we will study the convergence of the series

$$< F, \Phi_0 > \Phi_0 + \sum_{i=0}^{\infty} \sum_{l \in \Omega_{i+1}} a_l^{(i+1)} < F, \widetilde{\Psi}_{il} > \Psi_{il}.$$

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Set

$$\Lambda_{j,\omega}(F) =< F, \Phi_0 > \Phi_0 + \sum_{i=0}^{j-1} \sum_{l\in\Omega_{i+1}} a_l^{(i+1)} < F, \widetilde{\Psi}_{il} > \Psi_{il} + \sum_{l\in\omega} a_l^{(j+1)} < F, \widetilde{\Psi}_{jl} > \Psi_{jl}$$

where ω is a subset of Ω_{j+1}.

Main Theorem

For any $F \in C(S^d)$

$$\lim_{j \to \infty} \|F - \Lambda_{j,\omega}(F)\| = 0 \qquad (4)$$

First we will prove that the operators $\Lambda_{j,\omega} : C(S^d) \longrightarrow C(S^d)$, are uniformly bounded. We show that (4) holds on the set of spherical polynomials and our claim is a consequence of **Banach-Steinhaus** theorem. To prove the main theorem we need the following lemma:

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Lemma

For any $F \in C(S^d)$

$$< F, \Phi_0 > \Phi_0 + \sum_{i=0}^{j-1} \sum_{l \in \Omega_{i+1}} a_l^{(i+1)} < F, \widetilde{\Psi}_{il} > \Psi_{il}$$

$$= \sum_{l \in \Omega_j} a_l^{(j)} < F, \Phi_{jl} > \Phi_{jl}. \tag{5}$$

Construct a polynomial basis for $C(S^d)$.

- The following theorem due to **Krein-Milman-Rutman** plays the main role:
  Let $\{Q_k\}_{k=0}^{\infty}$ be a basis for a Banach space H and let $\alpha_k \in H^*$, $k = 0, 1, ...,$ be coefficient functionals for this basis. If $P_k \in H$ and $\|P_k - Q_k\| \leq \frac{2^{-k-2}}{\|\alpha_k\|} =: \lambda_k$ for all $k = 0, 1, ...,$ then the sequence $\{P_k\}_{k=0}^{\infty}$ is a basis for H.

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- If we can find a basis, say $\{Q_n\}$, for $C(S^d)$ and set $\lambda_k = \frac{2^{-k-2}}{\|\alpha_k\|}$ where $\{\alpha_k\}_{k=1}^{\infty}$ is the sequence of corresponding coefficient functional.
  So to construct a polynomial basis for $C(S^d)$ it's enough to construct a basis for it.

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- We start with finding an initial basis $\{Q_k\}$ for $C(S^d)$

Construct a polynomial basis for $C(S^d)$.

Step 1.

Counstruction a basis for the space $C([0,1]^d)$ and

$$C_0([0,1]^d) := \left\{ f \in C([0,1]^d) : f(x) = 0, \forall x \in \partial([0,1]^d) \right\}.$$

Let $\{f_n\}_{n=0}^{\infty}$ be the Faber-Cshaude basis for $C[0,1]$ defined by

$$f_0(x) = 1,\ x \in [0,1]$$

$$f_1(x) = x,\ x \in [0,1],$$

for $n = 2^k + i,\ k = 0, 1, ..., i = 1, 2, ..., 2^k,\ f_n$ is linear and continuous on $[\frac{i-1}{2^k}, \frac{2i-1}{2k+1}]$ and on $[\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}]$,

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin (\frac{i-1}{2^k}, \frac{i}{2^k}) \\ 1 & \text{if } x = \frac{2i-1}{2^{k+1}} \end{cases}$$

Construct a polynomial basis for $C(S^d)$.

Step 1.

The tensor product of d systems f_n is a basis for $C([0,1]^d)$, say $\mathcal{B} = \{F_k\}_{k=1}^{\infty}$. Elements of $\mathcal{B}$ are functions of the form

$$F(x) = f_{n_1}(x) f_{n_2}(x) \cdots f_{n_d}(x), x_j \in [0,1], n_j \in Z_+, j = 1, \cdots, d.$$

Denote by $\mathcal{B}$, the subset of $\mathcal{B}$ that contains of all the functions F such that $n_j \neq 0$, $n_j \neq 1$, $j = 1, \cdots, d$. Then $\mathcal{B}$, is a basis foe $C_0([0,1]^d)$.

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Denote by $\mathcal{B}'$ the subset of $\mathcal{B}$ that contains of all the functions F such that $n_j \neq 0$, $n_j \neq 1$, $j = 1, \cdots, d$. Then $\mathcal{B}'$ is a basis foe $C_0([0,1]^d)$.

Step 2.

Construction of basis for the space $C(B^d)$ and

$$C_0(B^d) := \left\{ f \in C(B^d) : f(x) = 0, \forall x \in \partial B^d \right\}.$$

By a change of variable, we can replace $[0,1]^d$ by $[-1,1]^d$, preserving the same notation $\mathcal{B}$, $\mathcal{B}'$ for the corresponding basis.

Construct a polynomial basis for $C(S^d)$.

Step 2.

Define a map

$$\phi : \mathbf{B^d} \longrightarrow [-1,1]^d$$

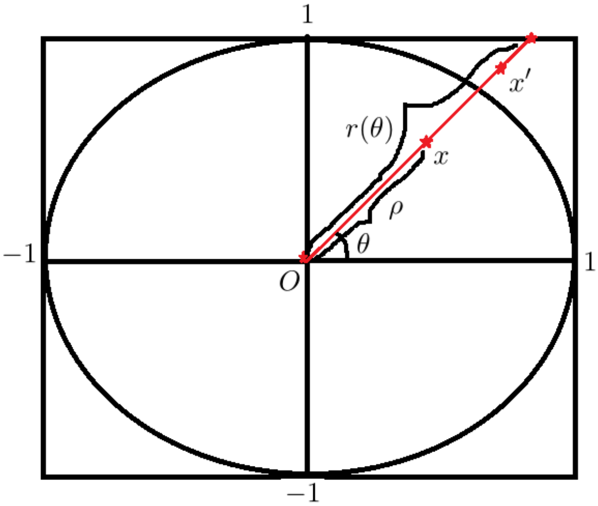
$$\phi(x) = (\rho r(\theta), \theta)$$

where $x = (\rho, \theta)$, $0 \le \rho \le 1$, $\theta \in S^{d-1}$ and $r(\theta)$ is the length of the segment

$$\left\{ y = (t, \theta) : t \ge 0, y \in [-1,1]^d \right\}.$$

Then θ is bijective and the functions $G_k := F_k(\phi)$, $k = 1, 2, ...$, constitute a basis for $C(B^d)$. Denote this basis by $\mathcal{B}'$.

A basis $\mathcal{B}'_{\prime} = \{G_k^{(0)}\}_{k=1}^{\infty}$ for $C_0(B^d)$ can be constructed similarly from $\mathcal{B}_{\prime}$.



$$\phi : \mathbf{B^d} \longrightarrow [-1, 1]^d$$

$$\phi(x) = (\rho r(\theta), \theta)$$

$$\rho \leq 1 \Rightarrow \rho r(\theta) \leq r(\theta)$$

Construct a polynomial basis for $C(S^d)$.

Step 3.

Construction of a basis for $C(S^d)$.
Let $C^{(o)}(S^d)$ and $C^{(e)(S^d)}$ be respectively the set of functions $f \in C(S^d)$ such that

$$f(x_1, x_2, ..., x_d, x_{d+1}) = f(x_1, x_2, ..., x_d, -x_{d+1})$$

and the set of functions $f \in C(S^d)$ such that

$$f(x_1, x_2, ..., x_d, x_{d+1}) = -f(x_1, x_2, ..., x_d, -x_{d+1}).$$

Construct a polynomial basis for $C(S^d)$.

Step 3.

Construction of a basis for $C(S^d)$.
Let $C^{(o)}(S^d)$ and $C^{(e)(S^d)}$ be respectively the set of functions $f \in C(S^d)$ such that

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and the set of functions $f \in C(S^d)$ such that

$$f(x_1, x_2, ..., x_d, x_{d+1}) = -f(x_1, x_2, ..., x_d, -x_{d+1}).$$

Each $f \in C(S^d)$ can be represented in the form $f = f^{(e)} + f^{(o)}$, where $f^{(e)} \in C^{(e)}(S^d)$, $f^{(o)} \in C^{(o)}(S^d)$. It is obvious that if $\{H_k^{(e)}\}_{k=1}^{\infty}$ and $\{H_k^{(o)}\}_{k=1}^{\infty}$ are bases for $C^{(e)}(S^d)$ and $C^{(o)}(S^d)$, then the system $\{H_k^{(e)}, H_k^{(o)}\}_{k=1}^{\infty}$ is a basis for $C(S^d)$.

Construct a polynomial basis for $C(S^d)$.

Step 3.

So it remaines to find bases for $C^{(e)(S^d)}$ and $C^{(o)}(S^d)$.
For $x = (x_1, x_2, ..., x_{d+1}) \in S^d$, set

$$H_k^{(e)}(x) = G_k(x_1, x_2, ..., x_d)$$

$$H_k^{(o)}(x) = \begin{cases} G_k^{(o)}(x_1, x_2, ..., x_d) & \text{if } x_{d+1} \geq 0 \\ -G_k^{(o)}(x_1, x_2, ..., x_d) & \text{if } x_{d+1} \leq 0 \end{cases}$$

$\{H_k^{(e)}\}_{k=1}^{\infty}$ and $\{H_k^{(o)}\}_{k=1}^{\infty}$ are the required bases.

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