Matrix coefficients of unitary representations and projections in $L^1(G)$

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Wavelet transform on $\mathbb{R}$

- Fix $\psi \in L^2(\mathbb{R})$ satisfying "admissibility condition"

$$\int_{\mathbb{R}\setminus\{0\}} \frac{|\widehat{\psi}(w)|^2}{|w|} dw = 1.$$

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- Equivalently, ψ is a wavelet for the representation π of the $ax + b$-group.

Wavelets

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π unitary representation of G.
For $\xi \in \mathcal{H}_\pi$, define $V_\xi : \mathcal{H}_\pi \to C(G)$, $V_\xi(\eta) = \langle \eta, \pi(\cdot)\xi \rangle$.

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- Every *irreducible* sub-representation of λ has wavelets.

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  - $\pi(xy) = \pi(x)\pi(y)$.
  - $\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$.
  - The map $G \to \mathcal{H}$, $x \mapsto \pi(x)\xi$ is continuous for every ξ.
- Matrix coefficient function $\phi : G \to \mathbb{C}$, $x \mapsto \langle \pi(x)\xi, \eta \rangle$.

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Theorem (Bochner's Theorem)

For Abelian locally compact group G, $B(G) = \mathrm{span}_{\mathbb{C}} P(G)$.

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Theorem

$(B(G), \|\cdot\|_B)$ with pointwise operations is a commutative unital Banach algebra.

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Example

- $\overline{C_c(G) \cap B(G)}^{\|\cdot\|_{B(G)}} = A_\lambda(G)$, with λ left regular rep'n.

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Fourier algebra $A(G) := A_\lambda(G) = \overline{B(G) \cap C_c(G)}^{\|\cdot\|_{B(G)}}$.

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Goal

- Construct projections in $L^1(G)$ using matrix coefficient functions.
- Identify all L^1-projections produced from certain subspaces of matrix coefficient functions.

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Unitary rep'n $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ extends to rep'n $L^1(G) \to \mathcal{B}(\mathcal{H}_\pi)$,

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Remark

Support of an L^1-projection is a compact open subset of $\hat{G}$.

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- $L^1(\mathbb{R})$ and $L^1(\mathbb{Z})$ do not have any nontrivial projections.

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Proof.

$$
\begin{aligned}
V_\xi(\xi) * V_\xi(\xi)(x) &= \int_G V_\xi(\xi)(y) V_\xi(\xi)(y^{-1}x)dy \\
&= \int_G \langle \xi, \pi(y)\xi \rangle \langle \xi, \pi(y^{-1}x)\xi \rangle dy \\
&= \langle V_\xi(\xi), V_\xi(\pi(x)\xi) \rangle_{L^2(G)} = \langle \xi, \pi(x)\xi \rangle.
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Can we characterize all minimal L^1-projections of G?

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Answer

Every minimal projection of $L^1(G)$ is of the form $\langle \xi, \pi(\cdot)\xi \rangle$ for some $\pi \in \hat{G}$.

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Example

- $G = \mathbb{R} \rtimes \mathbb{R}^+$. Then
  - $\lambda = \infty \cdot \pi_+ \oplus \infty \cdot \pi_-$.
  - $A(G) = A_{\pi_+} \oplus A_{\pi_-}$.

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Example

- $G = \mathrm{M}_2(\mathbb{R}) \rtimes \mathrm{GL}_2(\mathbb{R})$. Then
  - $\lambda = \infty \cdot \pi$.
  - $A(G) = A_\pi$.

Theorem (Alaghmandan-Gh.-Taylor)

G unimodular locally compact group.
u a projection in $L^1(G)$. Then,
$u \in A(G) \cap L^p(G)$ for every $1 \leq p \leq \infty$.