Weak and smooth synthesis sets of the Fourier algebras and their applicatons

Ebrahim Samei
University of Saskatchewan

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G: An n-dimensional Lie group

$\mathcal{D}(G)$: compactly supported C^∞ functions on G,

$A(G)$: The Fourier algebra of G.

$$I(E) = \{f \in A(G) \mid f(x) = 0 \text{ for all } x \in E\},$$

$$J(E) = \overline{\{a \in I(E) \mid \operatorname{supp} a \text{ is compact}\}},$$

$$J_{\mathcal{D}}(E) = \{f \in \mathcal{D}(G) \mid f(x) = 0 \text{ for all } x \in E\},$$

$$I_0(E) = \{f \in A(G) \cap C_c(G) \mid \operatorname{supp} f \cap E = \varnothing\}$$

E is:

- **a set of synthesis** if $I(E) = \overline{I_0(E)}$.
- **a set of local syn.** if $J(E) = \overline{I_0(E)}$.

- E is **a set of weak synthesis of degree at most** d if $I(E)^d = \overline{I_0(E)}$ for some positive integer d, i.e. if $I(E)/\overline{I_0(E)}$ is nilpotent of degree at most d.

- E is **a set of local weak synthesis of degree at most** d if $J(E)^d = \overline{I_0(E)}$ for some positive integer d, i.e. if $J(E)/\overline{I_0(E)}$ is nilpotent of degree at most d.

- E is **a set of smooth synthesis** if $J_{\mathcal{D}}(E) = I(E)$ and it is **a set of local smooth synthesis** if $J_{\mathcal{D}}(E) = J(E)$.

Theorem 1 *(***Ludwig-Turowska***)*

Let M be a smooth m-dimensional submanifold of G, and let $E \subseteq M$ be closed in G. Then:

(i) $J_{\mathcal{D}}(E)^{[\frac{m}{2}]+1} = \overline{I_0(E)}$.

(ii) If E is a set of smooth synthesis, then it is a set of weak synthesis of degree at most $[\frac{m}{2}] + 1$.

(iii) If E is a set of local smooth synthesis, then it is a set of local weak synthesis of degree at most $[\frac{m}{2}] + 1$.

Definition 2 *(***Domar, Kirsch-Müller, Ludwig-Turowska***)*

(i) *Any pair* $t \in G$ *and* $a \in \mathrm{Aut}(G)$ *give rise to a mapping* $\varphi : G \to G$, $\varphi(s) = a(ts)$, *which will be called an* **affine transformation** *of* G.

(ii) *A Lie group* H *is a* **group of affine transformation of** G *if* H *acts smoothly by affine transformations on* G. *Smoothly here means that the mapping* $H \times G \to G$, $(a, x) \mapsto a(x)$ *is smooth.*

(iii) *Let M be a smooth m-dimensional submanifold of G. A subset E of M has the* **cone property** *if the following holds:*

(1) E is closed in G.

(2) For every $x \in E$, there exists an open neighborhood U_x of $x \in G$ and a C^∞ mapping ψ_x from an open subset $W_x \subset \mathbb{R}^m$ containing 0 *into a Lie group of affine transformations H_x on G such that $\psi_x(0) = \mathrm{id}_G$ and there exists an open subset $W_x^0 \subset W_x$ such that:*

(2.1) 0 *is in the closure of* W_x^0.

(2.2) For every $y \in U_x \cap E$, $\psi_x(W_x^0)y$ *is contained in* E *and open in* M *and the mapping* $W_x^0 \to \psi_x(W_x^0)y$, $t \mapsto \psi_x(t)y$, *is a diffeomorphism.*

Theorem 3 *(***Ludwig-Turowska***)*
Let M *be a smooth* m*-dimensional submanifold of* G*, and let* $E \subseteq M$ *be a subset with the cone property. Then:*
(i) E *is a set of local smooth synthesis.*
(ii) E *is a set of local weak synthesis of degree at most* $[\frac{m}{2}] + 1$.

Anti-diagonal:

$$\check{\Delta}_G = \{(g, g^{-1}) \in G \times G \mid g \in G\}.$$

Forrest-S.-Spronk:

For G compact, $\check{\Delta}_G$ is a set of syn. for $A(G \times G)$ iff G_e is abelian.

Theorem 4 (**Park-S.**) (i) $\check{\Delta}_G$ *is a smooth manifold of dim.* n *with the cone prop.*

(ii) $\check{\Delta}_G$ *is a set of local weak syn. of degree at most* $[\frac{n}{2}] + 1$*.*

$$\psi : G \to \mathrm{Aff}(G \times G)$$

$$\psi(r)(g, h) = (gr, r^{-1}h).$$

Forrest-S.-Spronk: For G compact, $E = (\Delta_G \times \Delta_G)\Delta_{G\times G}$ is a set of syn. for $A(G^4)$ iff G_e is abelian.

Theorem 5 (**Park-S.**) (i) *E is a closed smooth submanifold of G^4 of dimension $3n$ with the cone property.*
(ii) *E is a set of loc. weak syn. for $A(G^4)$ of degree at most $[\frac{3n}{2}]+1$.*

$\psi : G^3 \to \mathrm{Aff}(G^4)$, $t = (p, q, r) \in G^3$

$$\psi(t)(g_1, g_2, g_3, g_4) =$$

$$(pg_1,\ pg_2qp^{-1},\ rg_3,\ rg_4qp^{-1}).$$

Connection with (operator) weak amenability of $A(G)$!

A Banach algebra A is **amenable** if for any Banach A-bimodule X, any bounded derivation $D : A \to X^*$ is inner (i.e. $\exists x^* \in X^*$ s.t. $D(a) = ax^* - x^*a$).

A is **weakly amenable** if any bounded derivation $D : A \to A^*$ is inner.

B. E. Johnson:
(i) $L^1(G)$ is amenable iff G is amenable.
(ii) $L^1(G)$ is weakly amenable.
(iii) $A(SU(2))$ is not even weakly amenable!

Ruan: $A(G)$ is operator amenable iff G is amenable.

Forrest-Runde:
$A(G)$ is amenable iff G is admits an abelian subgroup of finite index.

Spronk 02; Samei 05:
$A(G)$ is operator weakly amenable.

Missing part: When $A(G)$ is weakly amenable?

Theorem 6 *Let $E \subset G$ be a set of weak synthesis for $A(G)$. Then TFAE:*
(i) E is a set of synthesis;
(ii) E is essential (i.e $\overline{I^2(E)} = I(E)$);
(iii) $I(E)$ is operator weakly amenable.

Corollary 7 *Let G be compact, connected, and non-abelian. Then $I(E)$ is not operator weakly amenable for:*
(i) $E = \check{\Delta}_G$;
(ii) $E = (\Delta_G \times \Delta_G)\Delta_{G\times G}$;
*(iii) G is semisimple and E is the conjugacy class of any regular element in G (**C. Meaney**).*

A closed subset $E \subset H$ is a **Helson set** if $A(H)|_E = C_0(E)$.

Theorem 8 *Let H be an amenable, non-discrete group, and $E \subset H$ be a Helson-set of non-synthesis for $A(H)$. Then $\overline{I(E)^2} = I(E)$.*

*(***Bade-Curtis-Sinclair for H abelian***).*

Back to weak amenability of $A(G)$:

B. E. Johnson: $A(SU(2))$ is not weakly amenable.

Plymen: $A(G)$ is not weakly amenable for every compact connected non-abelian Lie group.

Forrest-Runde: If G_e is abelian, then $A(G)$ is weakly amenable.

Conjecture: The converse is also true!

Forrest-S.-Spronk: (i) For G maximally weakly almost periodic or SIN group, $A(G)$ is weakly amenable iff G_e is abelian iff $E = \check{\Delta}_G$ is a set of syn. for $A(G \times G)$.

(ii) For G compact, the map on

$$P : A(G \times G) \to A_\gamma(G) \ , \ u \times v \mapsto u * v$$

is a quotient map onto

$$A_\gamma(G) := \{f : \sum_{\pi \in \widehat{G}} d_\pi^2 \|\hat{f}(\pi)\|_1 < \infty\}.$$

Also the map $f \mapsto \tilde{f}(s,t) = f(st)$ is a left-inverse of P and is an isometric algebra isomorphism.

Choi-Ghandehari: $A(G)$ is not weakly amenable if G is a semisimple Lie group, a simply connected nilpotent Lie group, or the $ax+b$-group.

Key idea: Constructing a bounded derivation from $A(G) \to A(G)^*$ in the form of

$$(f,g) \mapsto \int_H \frac{\partial f(x)}{\partial t} g(x)dx, (f,g \in \mathcal{D}(G))$$

where the integration is taken over a suitable closed Lie subgroup H of G and the differentiation is taken over a one-parameter curve in H.

Alternative approach:

G: a connected Lie group.

• If $A(G)$ is weakly amenable, then $\check{\Delta}_G$ is a set of synthesis for $A(G \times G)$.

• If G is the reduced Heisenburg group or the $ax + b$-group, then $\check{\Delta}_G$ is a set of synthesis for $A(G \times G)$.

• Use the structure theory and synthesis property to extend the preceding result to other cases.

Theorem 9 *Suppose that G is a connected Lie group for which $\check{\Delta}_G$ is a set of local synthesis for $A(G \times G)$. Then:*

(i) G is solvable and does not have the $ax+b$-group as a closed subgroup;

(ii) if G is nilpotent, then G is abelian;

(iii) if $\check{\Delta}_L$ is not a set of local synthesis for $A(L \times L)$ for any 2-step solvable connected Lie group L, then G is abelian.