On old and new constructions of compact quantum groups

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The original motivation to study what is now called topological quantum groups was the desire to extend the classical Pontriagin duality for abelian groups to arbitrary groups. From the modern point of view theory of quantum groups belongs to the circle of ideas called noncommutative (or quantum) mathematics.

Noncommutative mathematics – general idea

The main idea behind the noncommutative mathematics is to

- first replace the study of a given space X by the study of an appropriate algebra of complex-valued functions on X (for example if X is a compact space, it is natural to consider $C(X)$, the algebra of continuous functions);
- characterize abstractly the algebras of that type: $C(X)$ for X compact – commutative unital C^*-algebras;
- finally drop the commutativity assumption:

Noncommutative (compact) topology – Theory of (unital) C^* – algebras.

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Compact quantum semigroups

A compact semigroup is a compact space G equipped with a continuous, associative map

$$\cdot : G \times G \to G.$$

On the level of the function algebras the map $\cdot$ induces a unital *-homomorphism

$$\Delta : C(G) \to C(G \times G) \approx C(G) \otimes C(G),$$

$$\Delta(f)(g_1, g_2) = f(g_1 \cdot g_2).$$

Definition

A unital C^*-algebra A is the algebra of continuous functions on a compact quantum semigroup if it admits a unital $*$-algebra homomorphism $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ such that

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \qquad \text{(coassociativity)}.$$

The map Δ is called a *coproduct* or a *comultiplication*.

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Towards the definition of compact quantum groups

How does one define then compact quantum groups? One could imitate what we did above with the multiplication for the inverse operation and the neutral element. This leads to the theory of *compact Kac algebras*. The alternative way is based on the following fact.

Proposition

A compact semigroup which satisfies the left and right cancellation rules is a compact group.

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Let us list some basic properties of and notions related to compact quantum groups:

- there exists a unique bi-invariant state, a so called Haar state $h \in C(\mathbb{G})^*$:

$$(h \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes h) \circ \Delta = h(\cdot)1_A.$$

- a finite-dimensional unitary representation of $\mathbb{G}$ is a unitary matrix $U \in M_n(C(\mathbb{G}))$ such that for $i, j = 1, \ldots, n$

$$\Delta(U_{ij}) = \sum_{k=1}^{n} U_{ik} \otimes U_{kj}.$$

- $C(\mathbb{G})$ admits a unique dense Hopf*-algebra $(\mathcal{A}, \Delta, \epsilon, S)$, spanned by the coefficients of the finite-dimensional unitary representations of $\mathbb{G}$ (i.e. U_{ij} above); in particular $\Delta : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$.

If a compact quantum group $\mathbb{G}$ admits $n \in \mathbb{N}$ and a unitary representation $U \in M_n(C(\mathbb{G}))$ such that $\{U_{ij} : i, j = 1, \ldots, n\}$ generates $C(\mathbb{G})$ as a C^*-algebra, then $\mathbb{G}$ is called a compact matrix quantum group and U a *fundamental unitary* representation.

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'Old' constructions of compact quantum groups

Compact quantum groups allow for some natural constructions (e.g. Cartesian products, quotients, etc.), and some more advanced methods of building new examples from given ones (e.g. the *bicrossed product construction*) but in the early years of the theory there were just two methods of constructing 'noncommutative' examples 'from scratch'.

Duals of discrete groups

If Γ is a discrete group, its quantum group dual $\hat{\Gamma}$ is defined via the identification

$$C(\hat{\Gamma}) := C^*(\Gamma),$$

where $C^*(\Gamma)$ denotes the full group C^*-algebra of Γ (i.e. the universal C^*-completion of the group ring $\mathbb{C}[\Gamma]$). The coproduct is given by the (continuous linear extension of)

$$\Delta(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma, \quad \gamma \in \Gamma.$$

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Deformations of classical compact Lie groups

Recall that $C(SU(2))$ is a commutative unital C^*-algebra generated by the functions $\alpha, \gamma : SU(2) \to \mathbb{C}$ such that

$$\alpha^*\alpha + \gamma^*\gamma = 1.$$

Group multiplication on $SU(2)$ induces on $C(SU(2))$ the coproduct

$$\Delta(\alpha) = \alpha \otimes \alpha - \gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Let $q \in [-1,1) \setminus \{0\}$. Define $C(SU_q(2))$ - unital C^*-algebra generated by operators α, γ such that:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma^*\gamma = 1,$$

$$\gamma^*\gamma = \gamma\gamma^*, \quad q\gamma\alpha = \alpha\gamma, \quad q\gamma^*\alpha = \alpha\gamma^*.$$

The coproduct making $SU_q(2)$ a compact quantum group is given by the formulas

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'New' constructions of compact quantum groups

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- quantum 'symmetry' and 'isometry' groups
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Quantum symmetry groups – general framework

Groups entered mathematics as collections of symmetries of a given object (a finite set, a figure on the plane, a manifold, a space of solutions of an equation).

How do we define a symmetry group of a given object X? We look at the family of 'all possible transformations' of X, usually preserving some given structure.

In modern language: we search for a universal object in the category of all groups acting on X.

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Actions

The action of a group G on a set X can be described as a map $\alpha : G \times X \to X$ satisfying certain natural conditions. In the quantum case we (as usual) 'invert the arrows':

Definition

Let $\mathbb{G}$ be a compact quantum group and let B be a unital C^*-algebra (think of B as $C(\mathbb{X})$). A map

$$\alpha : \mathsf{B} \to C(\mathbb{G}) \otimes \mathsf{B} \quad \alpha : C(\mathbb{X}) \to C(\mathbb{G}) \otimes C(\mathbb{X})$$

is called a (left, continuous) action of $\mathbb{G}$ on B if α is a unital *-homomorphism,

$$(\Delta \otimes \mathrm{id}_{\mathsf{B}}) \circ \alpha = (\mathrm{id}_{C(\mathbb{G})} \otimes \alpha) \circ \alpha$$

and additionally $\alpha(\mathsf{B})(C(\mathbb{G}) \otimes 1_{\mathsf{B}})$ is dense in $C(\mathbb{G}) \otimes \mathsf{B}$ (*Podleś/continuity condition*).

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Category of CQGs acting on a given C^*-algebra

Consider the category $\mathfrak{C}(\mathsf{B}) := \{(\mathbb{G}, \alpha)\}$ of compact quantum groups acting on a given C^*-algebra B. A morphism in the category $\mathfrak{C}(\mathsf{B})$:

$$\gamma : (\mathbb{G}_1, \alpha_1) \to (\mathbb{G}_2, \alpha_2)$$

is a unital *-homomorphism $\gamma : C(\mathbb{G}_2) \to C(\mathbb{G}_1)$ such that

$$(\gamma \otimes \gamma) \circ \Delta_2 = \Delta_1 \circ \gamma, \quad \alpha_1 = (\gamma \otimes \mathrm{id}_{\mathsf{B}}) \circ \alpha_2.$$

We say that the category $\mathfrak{C}(\mathsf{B})$ admits a universal (final) object, if there is $(\mathbb{G}_u, \alpha_u)$ in $\mathfrak{C}(\mathsf{B})$ such that for all $(\mathbb{G}, \alpha)$ in $\mathfrak{C}(\mathsf{B})$ there exists a unique morphism $\gamma : (\mathbb{G}, \alpha) \to (\mathbb{G}_u, \alpha_u)$. If such a universal object exists, it is unique.

$(\mathbb{G}_u, \alpha_u)$ – quantum symmetry group of B

Further we will also consider categories of actions preserving some 'extra' structure of B.

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Quantum permutation groups

Both classically and in the quantum framework the simplest symmetry groups are (quantum) permutation groups, which can be viewed as the universal (quantum) groups acting on a given finite set.

Theorem (S.Wang)

The category $\mathfrak{C}(\mathbb{C}^n)$ of quantum groups acting on the n-point set admits the universal object. It is denoted S_n^+ and called the quantum permutation group of an $n-$point set.

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S_n^+ is a compact matrix quantum group, its fundamental unitary is the n by n matrix whose entries are orthogonal projections (magic unitary):

$$U = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}, \qquad (q_{ij})^2 = q_{ij}^* = q_{ij}, \quad i, j = 1, \ldots, n.$$

Quantum symmetry group of M_n – negative result

Theorem (S.Wang)

The category $\mathfrak{C}(M_n)$ does not admit a universal object if $n > 1$.

The problem is related to the fact that there is a universal object in the category of compact quantum *semigroups* acting on M_n, but it is not a compact quantum *group*.

Quantum symmetry group of M_n – positive result

We say that the action α of $\mathbb{G}$ on B preserves a functional $\omega \in \mathsf{B}^*$ if

$$\forall_{b \in \mathsf{B}} \quad (\mathrm{id}_{C(\mathbb{G})} \otimes \omega)(\alpha(b)) = \omega(b) 1_{C(\mathbb{G})}.$$

Theorem (S.Wang)

Let D be a finite-dimensional C^*-algebra with a faithful state ω. The category $\mathfrak{C}(\mathsf{D}, \omega)$ of quantum groups acting on D and preserving the state ω admits a universal object.

Recently P. Sołtan showed that the universal compact quantum group in $\mathfrak{C}(M_2, \omega)$ is isomorphic to $SO_q(3)$ (with q dependent on the choice of ω).

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Let D be a finite-dimensional C^*-algebra with a faithful state ω. The category $\mathfrak{C}(\mathrm{D},\omega)$ of quantum groups acting on D and preserving the state ω admits a universal object.

Recently P. Sołtan showed that the universal compact quantum group in $\mathfrak{C}(M_2,\omega)$ is isomorphic to $SO_q(3)$ (with q dependent on the choice of ω).

Quantum symmetry group of M_n – positive result

We say that the action α of $\mathbb{G}$ on B preserves a functional $\omega \in \mathrm{B}^*$ if

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Beyond finite-dimensional cases – quantum isometry groups

As the last two results show, in general to establish the existence of a quantum symmetry group of some C^*-algebra B we need to put some more structure on B. This has led to the development of the theory of quantum isometry groups of noncommutative manifolds initiated by Goswami and further developed by Banica, Bhowmick, AS and others. We will present one example and a recent generalization.

Quantum isometry group of the dual of a finitely generated discrete group

Γ - finitely generated discrete group with (minimal, symmetric) generating set $S := \{\gamma_1, \cdots \gamma_n\}$

$l : \Gamma \to \mathbb{N}_0$ - word-length function

Theorem (J.Bhowmick +AS)

The category of all compact quantum groups acting on $C^*(\Gamma)$ and 'preserving the length' has a universal object; we call it the quantum isometry group of $\hat{\Gamma}$ and denote by $\mathrm{QISO}^+(\widehat{\Gamma})$.

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Theorem (Continued...)

$\mathrm{QISO}^+(\widehat{\Gamma})$ *is a compact matrix quantum group with a fundamental representation* $[q_{t,s}]_{t,s\in S}$, *where the elements* $\{q_{t,s} : t, s \in S\}$ *must satisfy the commutation relations implying that the prescription*

$$\alpha(\lambda_\gamma) = \sum_{\gamma' \in S : l(\gamma)=l(\gamma')} q_{\gamma',\gamma} \otimes \lambda_{\gamma'}, \quad \gamma \in \Gamma$$

defines (inductively) a unital **-homomorphism from* $C^*(\Gamma)$ *to* $C(\mathrm{QISO}^+(\widehat{\Gamma})) \otimes C^*(\Gamma)$.

Quantum symmetry groups of C^*-algebras equipped with orthogonal filtrations

This construction has now been generalized to quantum symmetry groups of C^*-algebras equipped with orthogonal filtrations.

Definition

An orthogonal filtration of a C^*-algebra equipped with a faithful state ω is a family $(V_i)_{i\in\mathcal{I}}$ of finite-dimensional subspaces of A such that

1. $V_0 = \mathbb{C}1_{\mathsf{A}}$;
2. for all $i, j \in \mathcal{I}$, $i \neq j$, $a \in V_i$ and $b \in V_j$ we have $\omega(a^* b) = 0$;
3. the set $\mathrm{Lin}(\bigcup_{i\in\mathcal{I}} V_i)$ is a dense *-subalgebra of A.

$\mathbb{G}$ acts on A in a filtration preserving way if there exists an action α of $\mathbb{G}$ on A such that the following condition holds:

$$\alpha(V_i) \subset V_i \odot C(\mathbb{G}), \quad i \in \mathcal{I}.$$

We will then write $(\alpha, \mathbb{G}) \in \mathfrak{C}_{\mathsf{A},\mathcal{V}}$.

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Existence result

Theorem

Let $(\mathrm{A}, \omega, (V_i)_{i\in\mathcal{I}})$ *be a* C^**-algebra with an orthogonal filtration. There exists a final object in the category* $\mathfrak{C}_{\mathrm{A},\mathcal{V}}$*; in other words there exists a universal compact quantum group* $\mathbb{G}_u$ *acting on* A *in a filtration preserving way. We call* $\mathbb{G}_u$ *the quantum symmetry group of* $(\mathrm{A}, \omega, (V_i)_{i\in\mathcal{I}})$.

Liberated/free quantum groups

In recent years T.Banica, B.Collins, S.Curran, R.Speicher (and others) have initiated the study of a so-called liberation procedure. The idea can be (very informally) described as follows

- consider your favourite compact group of matrices G
- find a presentation of $C(G)$ in terms of finitely many generators, preferably coefficients of a unitary representation
- 'liberate' the generators, that is drop the assumption that they must commute
- show that the resulting family of algebraic relations determines an algebra $C(\mathbb{G})$ for a certain compact quantum group $\mathbb{G}$. We usually write $\mathbb{G} = G^+$.

If $\mathbb{G}$ is a compact quantum group then the quotient of $C(\mathbb{G})$ by its commutator ideal is the algebra of functions on a certain compact group, which we will denote $\mathbb{G}_{clas}$ and call the classical version of $\mathbb{G}$.
The liberation procedure started from G should lead to a quantum group $\mathbb{G}$ such that $G = \mathbb{G}_{clas}$.

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Liberated/free quantum groups continued

The liberation procedure, though ill-defined, leads to many interesting connections between the theory of quantum groups and free probability. Up to last year it was most successful for 'real' compact groups. In particular categorical considerations lead naturally to four families of 'free' quantum groups, all defined via conditions on the entries of the fundamental representations:

- free quantum orthogonal group O_n^+ (entries in U selfadjoint);
- free quantum bistochastic group B_n^+ (entries in U selfadjoint and summing to 1 in each column)
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So – is there a connection?

We have seen that the quantum permutation group S_n^+ can be viewed on one hand as the quantum symmetry group of the n-point set, and on the other as the liberation of the classical permutation group S_n. Are there any more examples of that type?

Free groups

Theorem (Banica, Bhowmick, AS)

Consider $\mathbb{F}_n$ – the free group on n generators with the usual generating set. Then $\mathrm{QISO}^+(\widehat{\mathbb{F}_n})$ *can be described explicitly and turns out to be the liberation of the classical group* $\mathbb{T}^n \rtimes H_n$.

This might look mysterious, but

$$\mathbb{T}^n \rtimes H_n = \mathrm{ISO}(\mathbb{T}^n) = \mathrm{ISO}(\widehat{\mathbb{Z}^n})$$

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Further properties of $\mathrm{QISO}^+(\widehat{\mathbb{F}_n})$

Using a mixture of the free probabilistic quantum group techniques we can compute the representation theory of $\mathrm{QISO}^+(\widehat{\mathbb{F}_n})$, using the combinatorial language of (non-crossing) partitions; this in turn allows us to prove the results of the following type.

Theorem

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We sketched recent developments in the theories of quantum groups arising either as quantum symmetry groups or liberations of classical groups. Let us finish by listing examples of open problems:

- to what extent does $\mathrm{QISO}^+(\widehat{\Gamma})$ depend on the generating set S?
- can one axiomatize the liberation procedure (a possible approach is suggested by recent work of Banica, Speicher, Curran, and Collins, and also by Raum and Weber)?
- Kustermans and Vaes developed a satisfactory theory of locally compact quantum groups (admitting a complete duality). Can one construct examples of non-compact locally compact quantum groups as liberations or quantum symmetry groups?

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