p-variants of Fourier algebras on compact groups

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Fourier algebra

G – compact group
$\widehat{G}$ – cts. (unitary) dual, "elements" $\pi : G \to \mathrm{U}(d_\pi)$
$u \in \mathcal{C}(G)$, $\pi \in \widehat{G}$, Fourier coeff. matrix:

$$\hat{u}(\pi) = \int_G u(s)\pi(s^{-1})\, ds$$

Fourier algebra

$$\mathrm{A}(G) = \left\{ u \in \mathcal{C}(G) : \sum_{\pi \in \widehat{G}} d_\pi \|\hat{u}(\pi)\|_1 < \infty \right\}$$
$$\cong \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi \mathrm{S}^1_{d_\pi}$$

$\mathrm{A}(G)$ Banach function algebra; $\widehat{\mathrm{A}(G)} \cong G$.

Averaging over the diagonal

$\mathrm{A}(G) \otimes^{\gamma} \mathrm{A}(G)$ – proj've tens. prod.
$\mathrm{A}(G \times G) = \mathrm{A}(G)\hat{\otimes}\mathrm{A}(G)$ – op. proj've tens. prod.

$\Delta = \{(s,s) : s \in G\}$, $G \times G/\Delta \cong G$
$w : G \times G \to \mathbb{C}$, $\Gamma w(s) = \int_G w(sr, r)\, dr$
$\Gamma u \otimes v = u * \check{v}$ $(\check{v}(s) = v(s^{-1}))$

Fact

Γ(func'n alg. on $G \times G$) = func'n alg. on G

Theorem

(a) [Johnson '94] $\mathrm{A}_{\gamma}(G) = \Gamma(\mathrm{A}(G) \otimes^{\gamma} \mathrm{A}(G)) \cong \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^2 \mathrm{S}^1_{d_\pi}$
(b) [Forrest-Samei-S. '10]
$\mathrm{A}_{\Delta}(G) = \Gamma(\mathrm{A}(G \times G)) \cong \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^{3/2} \mathrm{S}^2_{d_\pi}$

What are $\mathrm{A}_\gamma(G)$, $\mathrm{A}_\Delta(G)$?

$\mathrm{A}_\gamma(G)$ is a Beurling-Fourier algebra with weight $\omega(\pi) = d_\pi$.
[Lee-Samei '12, Ludwig-S.-Turowska '12]

For $\mathrm{A}_\Delta(G)$, let's first determine its operator space structure.

Theorem [Rostami-S. '13]

Qua complete quotient of $\mathrm{A}(G \times G)$ via Γ,
$\mathrm{A}_\Delta(G) \cong \ell^1\text{-}\bigoplus_{\pi\in\widehat{G}} d_\pi^{3/2} \mathrm{S}^2_{d_\pi,R}$ (row spaces).

Theorem [Rostami-S. '13]

Qua complete quotient, $\Gamma(\mathrm{A}(G) \otimes^h \mathrm{A}(G)) \cong \ell^1\text{-}\bigoplus_{\pi\in\widehat{G}} d_\pi^{3/2} \mathrm{S}^2_{d_\pi,R}$.

p-variations of Fourier algebras

$\mathrm{A}(G) \cong \ell^1\text{-}\bigoplus_{\pi\in\widehat{G}} d_\pi \mathrm{S}^1_{d_\pi}$, $\mathrm{A}_\Delta(G) \cong \ell^1\text{-}\bigoplus_{\pi\in\widehat{G}} d_\pi^{3/2} \mathrm{S}^2_{d_\pi}$

Let $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$.
Can we build algebras with components $\mathrm{S}^p_{d_\pi}$?

$\mathbb{C}$-interpolation: $[\mathrm{S}^\infty_{d_\pi}, \mathrm{S}^1_{d_\pi}]_{1/p} = \mathrm{S}^p_{d_\pi}$
Use Pisier's interpolated operator space structure.

Theorem

$\mathrm{A}^p(G) \cong \ell^1\text{-}\bigoplus_{\pi\in\widehat{G}} d_\pi^{1+\frac{1}{p'}} \mathrm{S}^p_{d_\pi}$ defines a comp'ly cont've Ban. alg.

Theorem

$\widehat{\mathrm{A}^p(G)} \cong G$ and is regular on G.

$1 \le p \le q \le \infty \Rightarrow \mathrm{A}_\gamma(G) \subseteq \mathrm{A}^q(G) \subseteq \mathrm{A}^p(G)$ (cont'vely)

Other 2-variants of Fourier algebras

$\mathrm{A}_\Delta(G) \cong \ell^1\text{-}\bigoplus_{\pi\in\widehat{G}} d_\pi^{3/2}\mathrm{S}^2_{d_\pi,R}$, $\mathrm{A}^2(G) \cong \ell^1\text{-}\bigoplus_{\pi\in\widehat{G}} d_\pi^{3/2}\mathrm{S}^2_{d_\pi,OH}$

Op. space struct. on Hilbert spaces $\mathcal{H} \mapsto \mathcal{H}_E$ called

- *homogeneous* if $\mathcal{CB}(\mathcal{H}_E) = \mathcal{B}(\mathcal{H})$
- *subquadratic* if for projections $P_1 + \cdots + P_m = I$ on $\mathcal{H}$

$$\|[x_{ij}]\|^2_{\mathrm{M}_n(\mathcal{H}_E)} \leq \sum_{k=1}^{m} \|[P_k x_{ij}]\|^2_{\mathrm{M}_n(\mathcal{H}_E)}$$

- *subcross* if $\mathcal{H}_E \hat{\otimes} \mathcal{H}'_E \to (\mathcal{H} \otimes^2 \mathcal{H}')_E$ comp'ly cont'ly

Eg. OH, R, C, $R+C$, max all subcross; dual structures $OH = OH^*$, $R = C^*$, $C = R^*$, $R \cap C = (R+C)^*$, $\min = \max^*$ subquadratic

Theorem

$\mathcal{H} \mapsto \mathcal{H}_E$ subhomogeneous, subcross w. subquadratic dual $\Rightarrow$ $\mathrm{A}^2_E(G) \cong \ell^1\text{-}\bigoplus_{\pi\in\widehat{G}} d_\pi^{3/2}\mathrm{S}^2_{d_\pi,E}$ comp'ly cont've Ban. alg.

Weighted versions

[Lee-Samei '12, Ludwig-S.-Turowska '12]

Weight: $\omega : \widehat{G} \to \mathbb{R}^{>0}$

$$\omega(\sigma) \leq \omega(\pi)\omega(\pi') \text{ whenever } \sigma \subseteq \pi \otimes \pi'$$

Bounded below if $\inf_{\sigma \in \widehat{G}} \omega(\sigma) > 0$. Eg. ω symmetric: $\omega(\bar{\pi}) = \omega(\pi)$.

Dimension weights: $d^{\alpha}(\pi) = d_{\pi}^{\alpha}$ $(\alpha \geq 0)$

Theorem

$\mathrm{A}^p(G, \omega) \cong \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} \omega(\pi) d_{\pi}^{1+\frac{1}{p'}} \mathrm{S}^p_{d_{\pi}}$ comp'ly cont've Ban. alg.

$\mathrm{A}^p(G, d^\alpha)$ as an invariant of G

Theorem

$1 \leq p \leq \infty$, $p \neq 2$, $\alpha \geq 0$

$$\mathrm{A}^p(G, d^\alpha) \cong \mathrm{A}^p(H, d^\alpha) \text{ isometric isom'm } \Leftrightarrow G \cong H$$

Uses: extreme point argument & [Walter '72]

Remark

$\mathrm{A}^2(G, d^\alpha) \cong \mathrm{A}^2(H, d^\alpha)$ isom'c isom'm
$\Rightarrow \exists$ isomorphism $\mathrm{Trig}(G) \cong \mathrm{Trig}(H)$
bij'n $\varphi : \widehat{H} \to \widehat{G}$ so $\mathrm{Trig}_{\varphi(\pi)} \cong \mathrm{Trig}_\pi$.

Averaging over the diagonal (again)

Recall: $\Gamma w(s) = \int_G w(sr, r)\, dr$

Theorem

Let $\mathrm{A}^p_\Delta(G) = \Gamma(\mathrm{A}^p(G)\hat{\otimes}\mathrm{A}^p(G))$. Then we have isometric identification

$$\mathrm{A}^p_\Delta(G, d^\alpha) = \mathrm{A}^r(G, d^{\beta(p)+2\alpha})$$

$$\text{where } \frac{1}{r} + \frac{|p-2|}{2p} = 1 \text{ and } \beta(p) = \begin{cases} 4 - \frac{4}{p} & \text{if } 1 \leq p < 2 \\ \frac{1}{2} & \text{if } p \geq 2 \end{cases}$$

Proof uses $\mathcal{CB}(C^p_d, C^{p'}_d) = \mathrm{S}_d^{\frac{2p}{|2-p|}}$, isometrically (eg. [Xu '10]).

Compare with [Forrest-Samei-S. '10]:

$$\mathrm{A}^2_\Delta(G) = \mathrm{A}^1(G, d^2) \neq \mathrm{A}^2(G, d) = \mathrm{A}^2_{R,\Delta}(G)$$

Operator weak amenability

$\mathrm{A}^p(G)$ op. weakly am'ble if every c.b. derivation $D : \mathrm{A}^p(G) \to \mathrm{A}^p(G)^*$ is inner.

Theorem [Johnson '94, Plymen '01], [S. '02, Samei '06]

(a) $\mathrm{A}(G)$ w'kly am'ble for no conn'd non-ab'n Lie group.

(b) $\mathrm{A}(G)$ always op. weakly am'ble

Theorem

(a) $\mathrm{A}^p(\mathrm{SU}(2), d^\alpha)$ op. w'kly am'ble $\Leftrightarrow 1 \leq p < \frac{4}{3+2\alpha}$.

(b) G connected Lie, $\mathrm{A}^p(G, d^\alpha)$ not op. w'kly am'ble when $p \geq \frac{4}{3+2\alpha}$.

Uses: $\mathrm{A}^p_\Delta(G)$ admits point derivation $\Leftrightarrow p \geq \frac{4}{3+2\alpha}$.

Operator amenability

$\mathrm{A}^p(G)$ op. weakly am'ble if $\mathrm{A}^p(G, d^\alpha)\hat{\otimes}\mathrm{A}^p(G, d^\alpha)$ admits b.a.d.

Theorem [Johnson '94, Plymen '01], [Ruan '95]

(a) $\mathrm{A}(G)$ am'ble for no conn'd non-ab'n Lie group.

(b) $\mathrm{A}(G)$ always op. am'ble

We say G is <u>tall</u> if $\lim_{\pi\to\infty} d_\pi = \infty$.

Theorem

if $p > 1$, $\mathrm{A}^p(G)$ not op. amenable if G is

$$\text{tall; non-abelian connected Lie; or } = \prod_{n=1}^{\infty} F_n \text{ each } F_n \text{ finite.}$$

Uses: $\mathrm{A}^p_\Delta(G)$ dual space if $p > 1$, Γ(av'ed b.a.i) $\subset \mathrm{A}^p_\Delta(G)$ (tall); then restriction to direct product subgroups (modulo finite)

Restriction to a torus

[Herz '73, Takesaki-Tatsuuma '72]: $H \le G$, $\mathrm{A}(G)|_H = \mathrm{A}(H)$.

Theorem

If $p > 1$, $\mathrm{A}^p_{\mathrm{SU}(2)}(\mathbb{T}) = \mathrm{A}^p(\mathrm{SU}(2))|_{\mathbb{T}}$ is a predual of

$$\left\{ (a_k)_{k\in\mathbb{Z}} : \sup_{n\ge 0} \frac{1}{(n+1)^{1/p'}} \left(\sum_{j=0}^{n} |a_{n-2j}|^{p'} \right)^{1/p'} < \infty \right\}.$$

Assign $\mathrm{A}^p_{\mathrm{SU}(2)}(\mathbb{T})$ op. sp. struc. making $u \mapsto u|_{\mathbb{T}}$ c.q.

Theorem

(a) $\mathrm{A}^p_{\mathrm{SU}(2)}(\mathbb{T})$ (op.) w'kly am'ble $\Leftrightarrow 1 \le p < 2$.

(b) $\mathrm{A}^2_{\mathrm{SU}(2),R}(\mathbb{T})$ op. w'kly am'ble, but not w'kly am'ble

Mamnūnam!
(Thank you!)