Group operator systems

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To see that $C^*(G)$ exists, consider first the group algebra $\mathbb{C}[G]$ of G: it consists of all complex linear combinations $p = \sum_{\text{finite}} \lambda_g g$. Define a norm on $\mathbb{C}[G]$ by letting

$$\|p\| = \sup_{\pi} \| \sum \lambda_g \pi(g) \|,$$

and complete to obtain $C^*(G)$.

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$$\mathcal{B}(H)^+ = \{T \in \mathcal{B}(H) : T \geq 0\} \qquad ((T\xi, \xi) \geq 0, \ \forall \ \xi \in H).$$

$$M_n(\mathcal{B}(H)) = \mathcal{B}(H^n), \quad \text{hence, we have a cone } M_n(\mathcal{B}(H))^+.$$

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The family $(M_n(\mathcal{S})^+)_{n \in \mathbb{N}}$ of cones satisfies the condition

- $A \in M_{n,k}, X \in M_n(\mathcal{S})^+ \Rightarrow A^* X A \in M_k(\mathcal{S})^+$.

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Alternatively: operator systems are selfadjoint unital subspaces of C*-algebras.

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Stinespring's Theorem

If $\mathcal{A}$ is a C*-algebra and $\phi : \mathcal{A} \to \mathcal{B}(H)$ is a completely positive map then there exists a *-homomorsphism $\pi : \mathcal{A} \to \mathcal{B}(K)$ and an operator $V : H \to K$ such that

$$\phi(a) = V^* \pi(a) V, \qquad a \in \mathcal{A}.$$

Operator systems from discrete groups

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$$\mathcal{S}(\mathfrak{u}) = \mathrm{span}\{e, u, u^* : u \in \mathfrak{u}\} \subseteq C^*(G).$$

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The C*-envelope of an operator system $\mathcal{S}$ is a C*-algebra $C_e^*(\mathcal{S})$ together with an embedding $\iota : \mathcal{S} \to C_e^*(\mathcal{S})$ such that if $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a complete order embedding then there exists a *-epimorphism $\pi : C^*(\phi(\mathcal{S})) \to C_e^*(\mathcal{S})$ with $\pi(\phi(x)) = \iota(x)$, $x \in \mathcal{S}$.

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The C*-envelope of an operator system $\mathcal{S}$ is, heuristically, the smallest C*-algebra generated by $\mathcal{S}$.

A particular cases of interest: $\mathbb{F}_n$

$\mathbb{F}_n$: the free group on n generators $u_1, \dots, u_n$

The operator system of $\mathbb{F}_n$ will be denoted by $\mathcal{S}_n$. It does not depend on the choice of a particular set of n generators.

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The operator system $\mathcal{S}_n$ is characterised by the following universal property: *If $T_1, \ldots, T_n$ are contractions on a Hilbert space H then there exists a (unique) unital completely positive map $\phi : \mathcal{S}_n \to \mathcal{B}(H)$ such that $\phi(u_i) = T_i$, $i = 1, \ldots, n$.*

Idea of proof: dilate the contarctions to unitaries acting on a common Hilbert space, and use the universal property of $C^*(\mathbb{F}_n)$.

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Thus, $G = \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ (n copies) has elements of the form $h_{i_1} h_{i_2} \ldots h_{i_k}$, where h_i is the generator of the ith copy of $\mathbb{Z}_2$ in G, and $i_1 \neq i_2 \neq \cdots \neq i_k$.

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$NC(n) = \mathrm{span}\{1, h_1, \dots, h_n\}$: the operator system of G; it will be called the *operator system of the non-commutatuve n-cube*.

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Note that $NC(n) \subseteq C^*(G) = C^*(\mathbb{Z}_2) * \cdots * C^*(\mathbb{Z}_2)$. After Fourier transform, we may consider $NC(n)$ as the operator subsystem of $\ell_2^\infty * \cdots * \ell_2^\infty$ spanned by the copies of ℓ_2^∞.

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Note that there also exists an operator system of the commutative n-cube: the above universal property is fulfilled for pairwise commuting selfadjoint contractions.

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Note the relations $\sum_{i=1}^{c} e_{v,i} = 1$, for all $v = 1, \ldots, n$.

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α is *functorial* if

$\phi \in CP(\mathcal{S}, clS_1)$, $\psi \in CP(\mathcal{T}, \mathcal{T}_1) \Rightarrow$
$\phi \otimes \psi \in CP(\mathcal{S} \otimes_\alpha \mathcal{T}, \mathcal{S}_1 \otimes_\alpha \mathcal{T}_1)$.

Three tensor products

The *minimal* tensor product: if $\mathcal{S} \subseteq \mathcal{B}(H)$ and $\mathcal{T} \subseteq \mathcal{B}(K)$, represent $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{B}(H \otimes K)$; the resulting operator system is denoted $\mathcal{S} \otimes_{\min} \mathcal{T}$.

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The *maximal* tensor product $\mathcal{S} \otimes_{\max} \mathcal{T}$:

take $A \in M_k(\mathcal{S})^+$, $B \in M_m(\mathcal{T})^+$, $X \in M_{n,km}$; then $X(A \otimes B)X^*$ is a typical element of $M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+$.

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The *commuting* tensor product $\mathcal{S} \otimes_{\rm c} \mathcal{T}$:
$u \in (\mathcal{S} \otimes_{\rm c} \mathcal{T})^+$ if $(\phi \cdot \psi)(u) \geq 0$ whenever $\phi : \mathcal{S} \to \mathcal{B}(H)$, $\psi : \mathcal{T} \to \mathcal{B}(H)$ have commuting ranges.

Here $\phi \cdot \psi(x \otimes y) = \phi(x)\psi(y)$.

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$u \in (\mathcal{S} \otimes_{\rm c} \mathcal{T})^+$ if $(\phi \cdot \psi)(u) \geq 0$ whenever $\phi : \mathcal{S} \to \mathcal{B}(H)$, $\psi : \mathcal{T} \to \mathcal{B}(H)$ have commuting ranges.

Here $\phi \cdot \psi(x \otimes y) = \phi(x)\psi(y)$.

$\mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{S} \otimes_{\rm c} \mathcal{T} \to \mathcal{S} \otimes_{\min} \mathcal{T}$.

The Kirchberg Conjecture (KC)

This is a reformulation of the Connes Embedding Problem; it states that

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Theorem

The following are equivalent:
(i) (KC) holds true;
(ii) $\mathcal{S}_n \otimes_{\min} \mathcal{S}_m = \mathcal{S}_n \otimes_{\rm c} \mathcal{S}_m$ for every $n, m \geq 2$;
(iii) $\mathcal{S}_2 \otimes_{\min} \mathcal{S}_2 = \mathcal{S}_2 \otimes_{\rm c} \mathcal{S}_2$.

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Theorem

$\mathcal{S}_n \otimes_{c} \mathcal{S}_m \neq \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$. In fact, the identity is not 2-positive as a map from $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1$ into $\mathcal{S}_1 \otimes_{\max} \mathcal{S}_1$.

Note that $\mathcal{S}_1 = \mathrm{span}\{1, z, \bar{z}\}$, where z is the identity function on $\mathbb{T}$.

Further equivalences

Theorem

The following are equivalent:
(i) (KC) holds true;
(ii) $\mathcal{N}C(n) \otimes_{\min} NC(m) = NC(n) \otimes_{\rm c} NC(m)$ for every $n, m \geq 3$;
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Theorem (Tsirelson)

(ii) $\mathcal{N}C(n) \otimes_{\rm c} NC(m) \neq NC(n) \otimes_{\max} NC(m)$ for every $n, m \geq 2$.

Local experiments

Suppose that Alice and Bob perform an experiment in which Alice is given an input value x and produces an output value a, while Bob is given an input value y and produces an output value b. Assume that the possible values of x, y, a, b are 0 and 1.

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- $p^1_{a|x} \geq 0,\ p^2_{b|y} \geq 0$, for all $a, b, x, y \in \{0, 1\}$,
- $p^1_{0|x} + p^1_{1|x} = 1$ for $x = 0, 1$, and
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Let $\mathcal{V} = \{(u, v, w, t) : u + v = w + t\} \subseteq \ell^\infty_4$.

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- $p_{a,0|x,0} + p_{a,1|x,0} = p_{a,0|x,1} + p_{a,1|x,1} = p^1_{a|x}$, for all $a,x \in \{0,1\}$,
- $p_{0,b|0,y} + p_{1,b|0,y} = p_{0,b|1,y} + p_{1,b|1,y} = p^2_{b|y}$, for all $b,y \in \{0,1\}$.

Local boxes

A box $(p_{a,b|x,y})_{a,b,x,y}$ is *local* if there exist $(r(\lambda))_\lambda$ with $\sum_\lambda r(\lambda) = 1$ and, for each λ, elements

$$p^k(\lambda) = (p^k_{0|0}(\lambda), p^k_{1|0}(\lambda), p^k_{0|1}(\lambda), p^k_{1|1}(\lambda)) \in \mathcal{V}^+,$$

$p^k_{0|0}(\lambda) + p^k_{1|0}(\lambda) = 1$, $p^k_{0|0}(\lambda) + p^k_{1|0}(\lambda) = 1$ $k = 1, 2$, such that

$$p_{a,b|x,y} = \sum_\lambda r(\lambda) p^1_{a|x}(\lambda) p^2_{b|y}(\lambda), \quad a, b, x, y \in \{0, 1\}.$$

Quantum correlation boxes

Tsirelson in 1980 introduced *quantum* correlation boxes. These are the probability distributions $(p_{a,b|x,y})$ given by

$$p_{a,b|x,y} = Tr(\rho(A_x^a \otimes A_y^b)),$$

where A_x^a and A_x^b are positive operators acting on corresponding Hilbert spaces H_x and H_y such that $A_x^0 + A_x^1 = I$ and $A_y^0 + A_y^1 = I$ for all $x, y \in \{0, 1\}$, and ρ is a positive trace-class operator of unit trace.

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Let $\mathcal{P}$ be the set of all correlation boxes, $\mathcal{L}$ be the closure of the set of all local correlation boxes, and $\mathcal{Q}$ be the closure of the set of all quantum correlation boxes. Clearly, $\mathcal{L} \subseteq \mathcal{Q} \subseteq \mathcal{P}$ and each of these sets is convex.

Conection with non-commutative cubes

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- $\sum_{i=1}^{2} s(e_{v,i}) = 1$, $v = 1, 2$.

Thus, $(s(e_{v,i} \otimes e_{w,j}))_{v,i,w,j}$ is a box.

Connection with tensor products

Theorem

We have the following identities:

$$\mathcal{P} = \{(s(e_{v,i} \otimes e_{w,j})) \ : \ s \text{ is a state on } NC(2) \otimes_{\max} NC(2)\}$$

$$\mathcal{Q} = \{(s(e_{v,i} \otimes e_{w,j})) \ : \ s \text{ is a state on } NC(2) \otimes_{\min} NC(2)\}$$

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It follows that these three sets are pairwise distinct.

Quantum corelation boxes are studied for larger than 2 number of exepriments and a larger than 2 number of players: in this case one needs to involve the operator systems $\mathcal{S}(n, c)$, $c \geq 2$. There is hence a direct link with Kirchberg's Conjecture.

Chromatic numbers of graphs

Recall that a c-colouring of a graph $G = (V, E)$ is a map $r : V \to \{1, \ldots, c\}$ such that if $(v, w) \in E$ then $r(v) \neq r(v)$.
Smallest such c: the chromatic number $\chi(G)$.

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Let $D(n, c) = \mathbb{Z}_c \oplus \cdots \oplus \mathbb{Z}_c$ (n copies).
$C^*(D(n, c)) \cong \ell_c^\infty \otimes \cdots \otimes \ell_c^\infty \cong \ell^\infty(\Delta_{n,c})$, where $\Delta_{n,c} = \{1, \ldots, c\}^n$.

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Let $\mathcal{S}_{\min}(n, c)$ be the operator subsystem of $C^*(D(n, c))$ spanned by

$$(\delta_{i_1}, 0, \dots, 0), (0, \delta_{i_2}, 0, \dots, 0), \dots, (0, 0, \dots, \delta_{i_n}),$$

for $i_k = 1, \dots, c$, $k = 1, \dots, n$ (where $\mathbb{Z}_c = \{\delta_i : i = 1, \dots, c\}$).

Then $\mathcal{S}_{\min}(n, c) = \mathrm{span}\{e'_{v,i} : v \in V, 1 \leq i \leq c\}$, where $e'_{v,i}$ is the elementary tensor from $\ell_c^\infty \otimes \cdots \otimes \ell_c^\infty$ having all ones except for the v-th position, where it has the i-th element of the canonical basis of ℓ_c^∞.

The classical chromatic number via operator systems

Proposition

The chromatic number $\chi(G)$ of G is equal to the smallest $c \in \mathbb{N}$ for which there exists a state $s : \mathcal{S}_{\min}(n, c) \otimes_{\min} \mathcal{S}_{\min}(n, c) \to \mathbb{C}$ such that

$$\forall v, \forall i \neq j, s(e'_{v,i} \otimes e'_{v,j}) = 0,$$

$$\forall (v, w) \in E, \forall i, s(e'_{v,i} \otimes e'_{w,i}) = 0.$$

The quantum chromatic number $\chi_q(G)$

Cameron, Montanaro, Newman, Severini, Winter, 2007

The quantum chromatic number $\chi_{\mathrm{q}}(G)$

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A quantum c-colouring of G are two POVM's $(E_{v,i})_{i=1}^{c} \subseteq M_p$, $(F_{v,i})_{i=1}^{c} \subseteq M_q$ and a vector $\xi \in \mathbb{C}^p \otimes \mathbb{C}^q$ such that

$$\forall v, \forall i \neq j, \langle (E_{v,i} \otimes F_{v,j})\xi, \xi \rangle = 0,$$

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We have $\chi(G) \sim 2^N$, while $\chi_{\mathrm{q}}(G) = N$.

Further quantum versions

We can play the same game but allowing

- two infinite dimensional Hilbert spaces and tensors $E_{v,i} \otimes F_{w,j}$: $\chi_{\mathrm{qs}}(G)$;
- a single infinite dimensional Hilbert space and mutually commuting POVM's: $E_{v,i}F_{w,j} = F_{w,j}E_{v,i}$: $\chi_{\mathrm{qc}}(G)$.
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$\chi_{\rm qc}(G) \leq \chi_{\rm qmin}(G) \leq \chi_{\rm qs}(G) \leq \chi_{\rm q}(G) \leq \chi(G)$.

These quantum chromatic numbers can be expressed in terms of operator system tensor products...

Expression via tensor products

Theorem

- $\chi_{\mathrm{qc}}(G)$ is the smallest $c \in \mathbb{N}$ for which there exists a state $s : \mathcal{S}(n,c) \otimes_{\mathrm{c}} \mathcal{S}(n,c) \to \mathbb{C}$ such that

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The other chromatic numbers seem to be more promising. For example, to disrove Connes Embedding Problem, it suffices to exhibit a graph G with $\chi_{\mathrm{qc}}(G) < \chi_{\mathrm{qmin}}(G)$.

THANK YOU VERY MUCH!