The 3rd Seminar on
Combinatorial Commutative Algebra, February 5 and 6, 2014
School of Mathematics, IPM, Tehran

The Eventual Shape of the Betti Tables of Powers of Ideals

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Let G be an abelian group and S be a G-graded a Noetherian algebra over a commutative ring $A \subseteq S_0$. Let $I_1, \ldots, I_s$ be G-homogeneous ideals in S, and let M be a finitely generated G-graded S-module. We show that the shape of nonzero G-graded Betti numbers of $MI_1^{t_1} \ldots I_s^{t_s}$ exhibit an eventual linear behavior as the t_is get large.

This is based on a joint work with Marc Chardin and Huy Tài Hà.

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On Ideal Lattice Cryptography

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A lattice L is a set of points in the n-dimensional Euclidean space R^n with a strong periodicity property. Lattices were first studied by mathematicians Joseph Louis Lagrange and Carl Friedrich Gauss. Lattices have been used recently in computer algorithms and in cryptanalysis. Further *ideal lattices* are a new concept. Ideal lattices are lattices corresponding to ideals in rings of the form $\mathbb{Z}[x]/\langle f \rangle$ for some irreducible polynomial f of degree n.

One of the aims of cryptography is to protect information that is sent over an insecure channel. In 1976, Diffie and Hellman introduced the concept of public key cryptography, where cryptosystems are based on mathematically hard problems. Since then, several of such mathematical problems have been proposed as a basis for public key cryptography, with varying success. In 1996, Miklos Ajtai discovered that there are mathematical problems in the area of lattices that have some desirable properties with respect to cryptography. These problems are the Shortest Vector Problem (SVP) and the Closest Vector Problem (CVP). Since then, lattices have been used to construct several cryptosystems and other cryptographic applications. The main usefulness of the ideal lattices in cryptography stems from the fact that very efficient and practical collision resistant hash functions can be built based on the hardness of finding an approximate shortest vector in such lattices.

In this talk, lattices and ideal lattices are introduced and their connection with public key cryptography is considered to some extent.

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Configuration of Linear Subspaces in the Projective n Space

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There are many nice and interesting problems in algebraic varieties or schemes that can be reduced to a problem about a finite union of general linear subspaces. For example, in computing the dimension of the higher secant varieties of a variety, or in studying the nature of singularities of singular points of an algebraic variety (or scheme), the union of tangent spaces to the different branches of the variety at the singular points may brought more information about these points [3]. In the literature, these type of objects are known as linear subspace arrangements.
In this talk, by giving some specific examples from algebraic geometry, we concentrate on the geometric side of subspace arrangements and show how the notion of linear subspace arrangement arises in algebraic geometry in a natural way. Then show, how variations of the dimensions of the subspaces may complicate the problem.
There are other mathematical sources of subspace arrangements. For example, the excellent survey [1] describes major properties of subspace arrangements from a combinatorial viewpoint, and whenever considered as an $\mathbb{R}$-vector space, show that there are close relations with enumerative structures of subspace arrangement and the topology of the complement of the subspace.
To name other sources, linear subspace arrangement from a commutative algebra view point, have been studied extensively during last 30 years. Specifically, whenever subspaces are coordinate subspaces, the ideal of definition of the arrangement is just the Stanley-Reisner ideal of simplicial complex which is associated to the subspaces. For a survey of current state of knowledge about subspace arrangement from commutative algebra view point see [7].
More generally, topological problems, that are related to linear subspace arrangement have much delicate nature and have many applications in the real world [4].

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Asymptotic Behavior of Graded Betti Numbers of Powers of Ideals

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Using the concept of vector partition functions, we investigate the asymptotic behavior of graded Betti numbers of powers of homogeneous ideals in a polynomial ring over a field.

For a positive $\mathbb{Z}$-grading, our main result states that the Betti numbers of powers is encoded by finitely many polynomials. More precisely, $\mathbb{Z}^2$ can be splitted into a finite number of regions such that, in each of them, $\dim_k \left(\mathrm{Tor}_i^S(I^t, k)_\mu\right)$ is a quasi-polynomial in (μ, t). This refines, in a graded situation, the result of Kodiyalam on Betti numbers of powers in [18].

Our main statement treats the case of an arbitrary number of ideals in a $\mathbb{Z}^d$-graded algebra, for a positive grading, in the sense of [21].

This is a joint work with Amir Bagheri.

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A New Construction for Cohen-Macaulay Graphs

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Let G be a finite simple graph on a vertex set $V(G) = \{x_{11}, \ldots, x_{n1}\}$. Also let $m_1, \ldots, m_n \geq 2$ be integers and $G_1, \ldots, G_n$ be connected simple graphs on the vertex sets $V(G_i) = \{x_{i1}, \ldots, x_{im_i}\}$. In this paper, we provide necessary and sufficient conditions on $G_1, \ldots, G_n$ for which the graph obtained by attaching G_i to G is unmixed or vertex decomposable. Then we characterize Cohen–Macaulay and sequentially Cohen–Macaulay graphs obtained by attaching the cycle graphs or connected chordal graphs to an arbitrary graphs.

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Filtrations of Free Resolutions in Positive Dimension

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Two conjectures of M. Boij and J. Söderberg [1], proved by D. Eisenbud, G. Fløystad, F.O. Schreyer and J. Weyman [3], [4], state that the Betti diagram of a Cohen-Macaulay module M decomposes as a positive rational linear combination of pure diagrams. This linear combination is called the Boij-Söderberg decomposition of M. By a pure diagram we mean a Betti diagram of a module with exactly one non-zero entry in each column. This result nowadays known as Boij-Söderberg theory yields remarkable numerical information about graded free resolutions. As an application, this theory has led to a proof of the Herzog-Huneke-Srinivasan Multiplicity Conjecture. This conjecture gives lower and upper bounds for the multiplicity of a graded algebra in terms of the lowest and the highest degrees where the Betti numbers are non-zero.

A natural question is whether the decomposition of the Betti diagram of a module M into pure summands correspond to any sort of decomposition of the minimal free resolution of M. In [2], D. Eisenbud, D. Erman and F.O. Schreyer give sufficient conditions under which the Boij-Söderberg decomposition of a Betti diagram is reflected in the filtrations of the resolutions.

More precisely, Let $\sum_{t=0}^{s} c^t \pi(\mathbf{d}^t)$ be the Boij-Söderberg decomposition of Betti diagram of a CM module M over the polynomial ring $k[x_0, \ldots, x_n]$. Here $\pi(\mathbf{d}^t)$ is the Betti diagram of a module with pure free resolution

$$0 \to S(-d_n^t) \to \cdots \to S(-d_1^t) \to S(-d_0^t) \to 0,$$

associated to the degree sequence $\mathbf{d}^t = (d_0^t, d_1^t, \ldots, d_n^t)$. The question is when does the Betti diagram $c_0\pi(\mathbf{d}^0)$ correspond to the resolution of a submodule M' whose free resolution is term by term a summand of the free resolution of M?

In [2], they show that for a module of finite length, under sufficient conditions such a submodule M' exits. In this talk we investigate that to what extent this can be done for modules with the same type of resolutions but with positive Krull dimension.

This is a joint work with Gunnar Floystad.

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Minimal Free Resolution of Powers of Monomial Ideals

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Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over field $\mathbb{K}$. In this talk I will mention some results on minimal free resolution of powers of monomial ideals, essentially linear resolution of powers of edge ideals. After that, we present a criterion for componentwise linearity of powers of monomial ideals. In particular, we prove that if a square-free monomial ideal I contains no variables and some power of I is componentwise linear, then I satisfies gcd condition.

This talk is based on a joint work with N. Altafi, S.A.S. Fakhari and S. Yassemi.

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White's Conjecture is Preserved Under Taking the Expansion Functor

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In 1980, Neil White conjectured that the toric ideal associated to the basis of a matroid is generated by quadrics corresponding to symmetric exchanges. Later, in 2002, Jürgen Herzog and Takauki Hibi conjectured that this also holds for discrete polymatroids.

The aim of this talk is describing some algebraic and combinatorial behaviors of the expansion functor on discrete polymatroids. It is shown that, for a given discrete polymatroid P, the toric ideal of P is generated by quadrics corresponding to symmetric exchanges if and only if the toric ideal of any expansion of P has such property. This result, in a special case, says that White's conjecture is preserved under taking the expansion functor.

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Free Resolutions of Binomial Edge Ideals of Graphs

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The binomial edge ideal of a graph was independently introduced in [2] and [4], a few years ago. Let G be a finite simple graph with vertex set $[n]$ and edge set $E(G)$ and let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring over a field K. Then the **binomial edge ideal** of G in S, denoted by J_G, is generated by binomials $f_{ij} = x_i y_j - x_j y_i$, where $i < j$ and $\{i, j\} \in E(G)$. Since then, many authors studied these ideals from both combinatorial and algebraic points of view (see for example the following references). For instance, all graphs whose binomial edge ideals have quadratic Gröbner basis were characterized, according to the shape of the graph. Also, all graphs, for which the binomial edge ideals have linear resolutions, were classified. Some lower and upper bounds for the Castelnuovo-Mumford regularity were obtained in terms of the graphical parameters.

Here, we characterize all graphs whose binomial edge ideals have pure resolutions. Also, we study some graded Betti numbers of these ideals, with focus on the linear strand, with respect to graphical terms.

This is based on a joint work with Dariush Kiani.

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Filter-regular Sequences, Almost Complete Intersections and Stanley's Conjecture

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Let K be a field and $S = K[x_1, \ldots, x_n]$. Let I be a monomial ideal of S and $u_1, \ldots, u_r$ be monomials in S which form a filter-regular sequence on S/I. We show that S/I is pretty clean if and only if $S/(I, u_1, \ldots, u_r)$ is pretty clean. We also show that if either: 1) I is almost complete intersection, 2) I can be generated by less than four monomials; or 3) I is the Stanley-Reisner ideal of a locally complete intersection simplicial complex on $[n]$, then S/I is pretty clean and hence Stanley's conjecture holds for S/I in all of this cases.

This is a joint work with S. Bandari and K. Divaani-Aazar.

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Castelnuovo-Mumford Regularity under Reduction Processes on Square-free Ideals

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In this talk, we introduce "Castelnuovo-Mumford regularity" as one of the important invariants of graded modules. Then, we present the reduction processes on square-free monomial ideals in polynomial ring. More precisely, for a given square-free monomial ideal I in the polynomial ring, we will find a proper subset A of generators of I such that, the ideal generated by A has the same regularity as I. Finally, we will give some applications of these reduction processes.

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CM_t **Simplicial Complexes**

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Cohen-Macaulay-ness is a distinguished property in many areas of mathematics including algebraic geometry, commutative algebra and algebraic combinatorics. The role of Cohen-Macaulay property in algebraic combinatorics, initiated by Melvin Hochster, was culminated in the proof of the celebrated upper-bound conjecture by Richard Stanley. By a result of G. Reisner, a Cohen-Macaulay simplicial complex Δ is characterized by Cohen-Macaulay-ness of $\mathrm{link}_\Delta F$ for every face F in Δ (including the empty face). A simplicial complex Δ is Buchsbaum if and only if $\mathrm{link}_\Delta F$ is Cohen-Macaulay for every non-empty face F in Δ. The CM_t property unifies and naturally generalizes both Cohen-Macaulay and Buchsbaum properties. For a non-negative integer t a simplicial complex Δ is CM_t if it is pure and $\mathrm{link}_\Delta F$ is Cohen-Macaulay for all F is Δ with $|F| \geq t$, yielding Cohen-Macaulay-ness and Buchsbaum-ness for $t = 0, 1$, respectively.

In this talk, after recalling necessary preliminaries, some extensions of basic properties of Cohen-Macaulay and Buchsbaum simplicial complexes on CM_t simplicial complexes will be discussed. In particular, a characterization of bipartite CM_t graphs will be outlined as an extension of a result of Herzog and Hibi in the Cohen-Macaulay case.

This will cover some joint work with H. Haghighi and S. Yassemi.