The Problem of Principalization of Locally Monomial Ideal Sheaves

Razieh Ahmadian

IPM

This talk is based on my thesis under supervision of Prof. Cutkosky and Prof. Zaare,

and a joint work in progress with Dr. Raheleh Jafari.

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Statement of the Problem

The problem of **principalization** of $\mathcal{I}$ is to obtain a proper birational morphism $\lambda : \widetilde{X} \to X$, by means of a finite sequence of blowups with nonsingular centers, such that

- $\widetilde{X}$ is nonsingular, and
- $\mathcal{I}\mathcal{O}_{\widetilde{X}}$ is locally principal.

- The existence of principalization of an ideal sheaf on a nonsingular variety over an algebraically closed field of characteristic zero was first proven in
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Past Results and Recent Researches

One of the most interesting cases of the problem is the case of locally monomial ideal sheaf, i.e., when $\mathcal{I}_p$ is a monomial ideal for all $p \in X$. This case has many applications, in spite of its relative simplicity.

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Locally Monomial Ideal Sheaves

Definition

An ideal sheaf $\mathcal{I}$ on a nonsingular 3-fold X is **locally monomial** if for all $p \in X$ there exist regular parameters $x, y, z \in \mathcal{O}_{X,p}$ such that $\mathcal{I}_p$ is a monomial ideal in x, y, z.

Example

Suppose that there exist regular parameters $x, y, z \in \mathcal{O}_{X,p}$ such that the stalk of $\mathcal{I}$ at $p \in X$ is

$$\begin{aligned} \mathcal{I}_p &= (xy^2, x^3y, z^2) \\ &= (x^3, y^2, z^2) \cap (x, z^2) \cap (y, z^2). \end{aligned}$$

We observe that $\mathrm{V}(\mathcal{I}_p) = \mathrm{V}(\sqrt{\mathcal{I}_p})$ is the union of two coordinate curves (with local equations $x = z = 0$ and $y = z = 0$ at p).

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The Geometry of Locally Monomial Ideal Sheaves

Definition

1. Suppose that X is a nonsingular 3-fold. Let $p \in X$ and let x, y, z be regular parameters in $\mathcal{O}_{X,p}$. A **coordinate subvariety** is a subvariety of $\mathrm{Spec}\mathcal{O}_{X,p}$ defined by setting some subset of the parameters x, y, z equal to zero.
2. A reduced subscheme $T \subset X$ is called **simple normal crossing (SNC)** if for every closed point $p \in T$, there are regular parameters x, y, z in $\mathcal{O}_{X,p}$ such that $T \cap \mathrm{Spec}\mathcal{O}_{X,p}$ is a finite union of coordinate subvarieties of $\mathrm{Spec}\mathcal{O}_{X,p}$.

A Basic Fact

Let $\mathcal{I}$ be a locally monomial ideal sheaf on a nonsingular 3-fold X. Then $\mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$ is SNC.

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$$\mathbf{W}_{\mathcal{I}}(\mathbf{X}) = \{\mathbf{p} \in \mathbf{X} \mid \mathcal{I}_{\mathbf{p}} \text{ is not principal}\}.$$

Proposition

Suppose that X is a nonsingular 3-fold, and $\mathcal{I}$ is a locally monomial ideal sheaf on X. Then $W_{\mathcal{I}}(X)$ is SNC.

The proof is based on these facts:

- Since X is nonsingular there exists a factorization $\mathcal{I} = \mathcal{J}\mathcal{N}$ where $\mathcal{J}$ is an invertible ideal sheaf on X, and $\mathcal{N}$ is the defining ideal sheaf of $W_{\mathcal{I}}(X)$.
- As $\mathcal{I}$ is a locally monomial ideal sheaf, so are $\mathcal{J}$ and $\mathcal{N}$.

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The Invariant Order

Definition

- Let $(R, \mathfrak{m})$ be a regular local ring. Suppose that $J \subset R$ is an ideal. The **order** of J in R is

$$\nu_R(J) = \max\{k \mid J \subset \mathfrak{m}^k\}.$$

- Suppose that p is a point on a nonsingular variety X and $\mathcal{J} \subset \mathcal{O}_X$ is an ideal sheaf. We denote

$$\nu_p(\mathcal{J}) = \nu_{\mathcal{O}_{X,p}}(\mathcal{J}\mathcal{O}_{X,p}) = \nu_{\mathcal{O}_{X,p}}(\mathcal{J}_p).$$

If $V \subset X$ is a subvariety, we denote

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Definition (Principalization Sequence)

Let $\mathcal{I}$ be a locally monomial ideal sheaf on a nonsingular 3-fold X. Then we have $\mathcal{I} = \mathcal{J}\mathcal{N}$ where $\mathcal{J}$ is an invertible ideal sheaf, and $W_{\mathcal{I}}(X) = \mathrm{Supp}(\mathcal{O}_X/\mathcal{N})$. A **principalization sequence** of $\mathcal{I}$ is a sequence of blowups

$$\cdots \to X_n \xrightarrow{\lambda_n} X_{n-1} \cdots \to X_i \xrightarrow{\lambda_i} X_{i-1} \to \cdots X_1 \xrightarrow{\lambda_1} X$$

such that each $\lambda_i : X_i \to X_{i-1}$ has a nonsingular center T_{i-1} satisfying the following conditions:

1. For all $i \geqslant 0$, $\mathcal{I}_i$ is locally monomial, $W_{\mathcal{I}_i}(X_i)$ is SNC and $T_i \subseteq W_{\mathcal{I}_i}(X_i)$ where $\mathcal{I}_i = \mathcal{I}\mathcal{O}_{X_i}$.
2. Let $\mathcal{N}_i = \mathcal{N}\mathcal{O}_{X_i}$ be the transform of $\mathcal{N} = \mathcal{N}_0$ on X_i, so that $\mathrm{Supp}(\mathcal{O}_{X_i}/\mathcal{N}_i) = W_{\mathcal{I}_i}(X_i)$. Let

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Definition (Permissible Center)

$$r_i = \max\{\nu_p(\mathcal{N}_i) \mid p \in X_i\}$$

where $\nu_p(\mathcal{N}_i)$ is the order of $(\mathcal{N}_i)_p$ in $\mathcal{O}_{X_i,p}$.
We require that for all i, the center T_i is an irreducible component of maximal dimension of

$$\operatorname{Max} W_{\mathcal{I}_i}(X_i) = \{p \in X_i \mid \nu_p(\mathcal{N}_i) = r_i\}.$$

Theorem (Finiteness of the Principalization Sequences)

Suppose that X is a nonsingular 3-fold and $\mathcal{I}$ is a locally monomial ideal sheaf on X. Then any principalization sequence of $\mathcal{I}$ terminates after a finite number $n \geqslant 0$ of blow ups with $W_{\mathcal{I}_n}(X_n) = \emptyset$. Furthermore, $\mathcal{I}\mathcal{O}_{X_n}$ is a locally principal and locally monomial ideal sheaf on X_n.

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Idea of the Proof

- The main part of the theorem is the finiteness of a principalization sequence, and to prove this, we have used **proof by contradiction.**
- We take a **hypersurface of maximal contact**? reducing to dimension 2, and then we use the Abhyankar's proof of principalization of ideal sheaves in nonsingular varieties of dimension 2 to derive a contradiction.

Example

Suppose that there exist regular parameters $x, y, z \in \mathcal{O}_{X,p}$ such that the stalk of $\mathcal{I}$ at $p \in X$ is

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Some Techniques of Monomial Ideals

We have some ideas of defining new invariants using the primary decomposition of the monomial stalk $\mathcal{I}_p$ of the locally monomial ideal sheaf $\mathcal{I}$ at $p \in X$.

Definition

- Let $(R, \mathfrak{m})$ be a regular local ring and $J \subset R$ be a monomial ideal in R. Suppose that $J = \bigcap_{j=1}^{k} Q_j$ is the unique presentation of J as an irredundant intersection of irreducible monomial ideals, where each Q_j is generated by pure powers of the variables. Let $\mathbf{Ass}(J) = \{P_j := \sqrt{Q_j} \mid Q_j \text{ apears in the presentation } \bigcap_{j=1}^{k} Q_j\}$. We define

$$\hbar_R(J) := \max\{height(P_j) \mid P_j \in \mathbf{Ass}(J)\}.$$

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$$\hbar_p(\mathcal{I}) := \hbar_{\mathcal{O}_{X,p}}(\mathcal{I}\mathcal{O}_{X,p}) = \hbar_{\mathcal{O}_{X,p}}(\mathcal{I}_p).$$

A Basic Fact

Suppose that $\mathcal{I}$ is a locally monomial ideal sheaf on a nonsingular variety X. We note that

$$\mathcal{I} \text{ is locally principal } \Leftrightarrow \hbar_p(\mathcal{I}) = 1 \text{ for all } p \in X.$$

- This motivates us to consider this invariant for constructing a principalization algorithm. We have some proved evidence that $\hbar$ is a good choice, although it needs a complement.

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Example

Let's consider the following stalk of a locally monomial ideal sheaf $\mathcal{I}$ at a point p in a nonsingular 3-fold X. Suppose that there exist regular parameters $x, y, z \in \mathcal{O}_{X,p}$ such that

$$\begin{aligned}\mathcal{I}_p &= (x^2y^4, x^2y^3z, x^2z^2, y^4z) \\ &= (x^2, y^4) \cap (x^2, z) \cap (y^3, z^2) \cap (y^4, z).\end{aligned}$$

Clearly, $\hbar_p(\mathcal{I}) = 2$. Suppose that $\pi_1 : X_1 \to X$ is the blow up of X at the nonsingular curve C with local equations $x = y = 0$ at p. Then at each point $\tilde{p} \in \pi_1^{-1}(p)$, we have regular parameters $\tilde{x}, \tilde{y}, \tilde{z}$ such that $\tilde{p}$ can be expressed (uniquely) in one of the forms:

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Example

$\mathfrak{cb}1.$ $x = \tilde{x}, y = \tilde{x}(\tilde{y} + \alpha), z = \tilde{z}$, with $\alpha \in \mathfrak{k}$, or

$\mathfrak{cb}2.$ $x = \tilde{x}\tilde{y}, y = \tilde{y}, z = \tilde{z}$.

If ($\mathfrak{cb}1$) holds at p with $\alpha = 0$ (for simplicity), then

$$\mathcal{I}_{1,\tilde{p}} = (x^6y^4, x^5y^3z, x^2z^2, x^4y^4z) = (x^2)(x^4y^4, x^3y^3z, x^2y^4z, z^2),$$

and $\mathcal{N}_1 = (x^4y^4, x^3y^3z, x^2y^4z, z^2)$ has the primary decomposition

$$(x^3, y^4, z^2) \cap (x^4, y^3, z^2) \cap (x^4, z) \cap (x^2, z^2) \cap (y^3, z^2) \cap (y^4, z).$$

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